from

the **AMERICAN PEOPLE**

THE
FLOW OF HEAT
IN METALS

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A SERIES OF FIVE EDUCA-
TIONAL LECTURES ON
THE FLOW OF HEAT IN
METALS PRESENTED TO
MEMBERS OF THE AMERI-
CAN SOCIETY FOR METALS
AT THE TWENTY-THIRD
NATIONAL METAL CON-
GRESS AND EXPOSITION,
PHILADELPHIA, PA.,
OCTOBER 20-25, 1941

FOREWORD

A PARADOX familiar to workers in science is that as the amount of our knowledge increases the statement of that knowledge can be made more brief; for science is organized knowledge and the greater the knowledge, the better the organization — and the greater the organization, the more complete and concise become the principles and laws which represent the essence of the experimental observations. Further brevity is also achieved by a translation of these principles and laws into mathematics, which is one of the most concise of languages. A striking illustration of this paradox occurs in the field of thermodynamics in its original sense of heat flow. The basic concepts describing the conduction of heat were succinctly and elegantly set forth over a century ago by Fourier with a degree of completeness which has been difficult to improve on, since his work directly suggested or foreshadowed most of the later developments. In few other fields of science has such condensation been achieved. In consequence, all the essential information regarding the flow of heat in metals could probably be given by a single page of equations accompanied by several tables of physical constants.

A treatise of this kind, although intellectually satisfying to the pure scientist, has, however, a distinctly limited value to those concerned with applications of science, who are commonly less fluent in the language of mathematics; hence, if this vast store of information is to be useful to the technologist it must be made available in the technologist's language, even at the expense of concise expression. The loss of brevity attendant upon avoidance of mathematics is not without its compensations, however, for the formal severity of mathematics has its disadvantages, as has been pointed out by Lewis and Randall:

"Mathematics offers a wonderful shorthand for the pre-

cise formulation of well standardized ideas. On the other hand, the expressions of mathematics are lacking in humor, which is to say that they are no suitable medium for those finer shades of thought which are often necessary in the exposition of ideas which are on the way towards standardization. ....It is a dangerous thing to use any kind of a mathematical equation unless we keep its meaning before us and we are able to express this meaning without the symbolism which mathematics affords."

In these lectures an effort has been made to emphasize the meaning of a symbol rather than the symbol itself, without, however, consciously sacrificing any desirable element of mathematical rigor. Occasional sentences in the language of mathematics have crept in but their use is justified on the same basis as is the incorporation into English of a foreign word such as "Blitzkrieg"; they are simple, commonly understood, and express concisely a whole set of ideas.

A writer on a scientific subject must always envy the novelist who is not obliged to coerce his story into some artificial and allegedly logical order. To arrange in serial order a great mass of branching and interlocking ideas is always a difficult task. One is continually confronted by a choice of evils and unless one possess the literary genius of a Wodehouse in organizing and treating parallel situations the many ramifications of the subject are likely to get completely out of hand. In the case of lectures of this sort, there is a dual problem in that what appears the best sequence for a printed text does not usually fit in well with the requirements of five lectures of approximately equal length. The published text, therefore, of necessity differs somewhat from the lectures as given.

The reader is warned that no attempt has been made to review critically the many published data on the thermal conductivity of metals. Such a task is a formidable one and is

properly the function of these organizations whose purpose is to publish tables of physical constants. Instead, the aim has been to review principles rather than isolated measurements and to evolve a general framework which will stand substantially unaltered even though the details provided by individual data may change. In line with this policy, since we are concerned chiefly with comparisons, a set of observations which establishes a given trend is used as it is reported, without any attempt to adjust the data to what may be considered better absolute values. This results in the appearance of a number of significantly different values for the conductivity of the same metal. In defense of such apparent discrepancies it is necessary only to recall that the determination of thermal conductivity is, in general, not a very accurate measurement, an accuracy of 10 to 20 per cent being about all that can be expected in many cases, whereas for some of the older measurements even this accuracy is not approached. On the other hand, the precision and reproducibility of such determinations are generally much better than the accuracy, so that the results obtained on a comparable set of specimens tested by the same method are commonly good enough to establish a trend beyond question. In view of these facts, the appearance of different values for the thermal conductivity of the same metal does not alter the fundamental pattern of behavior any more than the fact that errors are constantly being made in numerical calculations need diminish our confidence in the principles of arithmetic.

In preparing these lectures I have received friendly assistance from a number of sources, notably from the Aluminum Company of America, The Dow Chemical Company and the International Nickel Company, who supplied data on a number of materials. I am particularly indebted to Handy and Harman, who supplied the silver cylinder used in making the tests described in Chapter IV, to Dr. E. C. Bain for his con-

structive comments, to Dr. M. A. Grossmann for many helpful suggestions and for some of the data presented, and finally to Dr. John Johnston for his continued interest and stimulating discussion.

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Kearny, N. J.
April, 1942

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Chapter I

THE NATURE OF HEAT AND OF METALLIC CONDUCTION

O F ALL scientific tools, words are the most widely used and the most important; they are used in all sciences, they are the medium in which we express our ideas and communicate them to others, and, since most people think in terms of words, they have an influence on our habits of thinking and even on our logic. They are, however, at best but an imperfect tool for they lack flexibility and precision; a given word remains a fixed combination of letters incapable of following the subtle changes in meaning which are always taking place. An ideally precise language would require an almost infinite number of words, one for each shade of meaning, but such a language would be cumbersome and quite impractical for ordinary use for which a limited vocabulary must suffice. It is necessary, therefore, to treat words with the same care and respect with which other scientific instruments are handled, to check them constantly, and to keep their meanings carefully calibrated, preferably in terms of operations.

Our first task, therefore, is to define the terms in which our fundamental concepts are expressed and there is no better place to begin than with the title of these lectures, namely, "The Flow of Heat in Metals". This phrase is concise and seemingly definite; at least it conveys to most readers a fairly exact idea of what is to be discussed. Yet it is a fact that the three significant words in this title, namely, "Flow", "Heat", and "Metals", are ambiguous and two of them, in the sense in which we shall use them, represent no physical reality.

Take first the word "metal". What exactly is a metal? The chemist views a metal as an element which forms basic

compounds, a substance which reacts with acids to form salts. On this basis, ammonia is a metal, yet it is entirely outside the province of these lectures. Similarly, the physicist may say that a metal is a substance which is a good conductor of electricity, yet this definition includes certain oxides and sulphides, which again are outside our scope. The metallurgist is likely to think of a metal as a solid element, or mixture of elements, which has a characteristic lustre, is a good conductor of heat and electricity, and possesses a certain strength, or ductility, or hardness, as the case may be, which make it uniquely useful as a material of construction. This statement, though admittedly vague, is, for our present purpose, as good a definition as any. We shall, therefore, confine this discussion almost entirely to those elements and alloys which are used as materials of construction and are produced commercially in reasonably large quantities.

Take next the word "heat". What is heat? Stripped to its essentials, the answer is that at present heat energy must be regarded as an exceedingly interesting but relatively recent invention; a theoretical abstraction which is more talked about than understood. If such a statement seems fantastic, it is only necessary to recall that no one has seen, felt, heard, tasted or smelled this mysterious something called heat. What we do see and feel is a change in temperature and we have invented a cause for this change in order to fill a gap in our logic. Since heat is an abstraction it is natural that its mo··t concise and elegant description is made in the language of abstractions, namely, mathematics. It is therefore neither by accident, nor by desire of authors to impress their readers, that a treatise on heat fairly bristles with equations; on the contrary, it is rather a matter of succinctness of expression.

As to the third term in our subject, namely "flow", it is obvious that since heat is merely a concept and is therefore intangible, there can be no flow in the sense of a transfer of

a fluid from one place to another. We are, therefore, faced by the fact that there attaches to our subject a considerable degree of indefiniteness and it is necessary to look a little more carefully into what we mean by these loose, though unquestionably useful, terms, and to see how we came to use them.

NATURE OF HEAT

From the time that Man began to speculate as to the cause of the sensation of heat down to about a century ago, two rival views were generally favored to the virtual exclusion of all others. The first which was probably the older, for it went back at least as far as Plato, but which contained, nevertheless, the germ of the modern theory, postulates that heat is due to the rapid vibration of the particles of which matter is composed; that is, that heat is essentially motion. The adherents to this hypothesis, who included among their number such careful thinkers as Francis Bacon, seem, however, always to have been decidedly in the minority. The second, and more widely accepted doctrine, held that the sensation of heat was associated with the presence of a subtle, all-pervading, elastic fluid called caloric, which, though self-repellent, was attracted to particles of matter, the strength of the attraction varying with the nature of the matter. Caloric was believed to be indestructible and uncreatable; it was considered to be abundant in a warm body but lacking in a cold one; it was supposed to flow from a place of high temperature to one of low temperature, as a liquid flows from a high to a low level. This view, on the basis of which it was possible to predict correctly the final temperature attained when a system of bodies initially at different temperatures was brought to thermal equilibrium, appeared to be adequate for all purposes until quantitative measurements were made of the generation of heat by friction, when, in spite of a number of ingenious modifications, it was no longer

able to explain the experimental data. In 1798, Benjamin Thompson, Count Rumford, an American forced into exile in Europe because of his lack of sympathy with the Revolutionists, reported to the Royal Society of London the results of some controlled experiments on the generation of heat in the boring of cannon, which demonstrated beyond question that the amount of heat produced in boring was apparently inexhaustible and could not possibly be accounted for in terms of an uncreatable fluid. The conclusion which Rumford drew was inescapable. To quote his own words: "It is hardly necessary to add that anything which any insulated body or system of bodies can continue to furnish without limitation cannot possibly be a material substance; and it appears to me to be extremely difficult, if not quite impossible, to form any distinct idea of anything capable of being excited and communicated in these experiments, except it be MOTION." As in the case of many scientific advances, the significance of this demonstration was not at once realized, nor indeed was Rumford's conclusion universally accepted until nearly fifty years later, and during that period a number of exceedingly important advances were made on the basis of the outmoded caloric theory; notable among these were Fourier's quantitative treatment of heat flow and Carnot's work on heat cycles. Moreover, the views of the caloric theory, particularly the belief that caloric was indestructible and uncreatable, undoubtedly influenced Robert Mayer in his formulation of the law of conservation of energy.

By the time that the more open-minded had accepted Rumford's demonstration that heat is motion and had begun to inquire precisely what kind of motion was concerned, the atomic hypothesis and the kinetic theory of matter had been developed to a point where they supplied an answer. Heat was recognized as a manifestation of molecular motion and the temperature of a body was recognized as a statistical

measure of the velocity of the motion of the atoms or mole-
cules of which the body is composed. On this basis a change
in temperature of a body corresponds to a change in the mean
velocity of vibration of its molecules.

But Rumford's observations included something beyond
a demonstration that heat is a mode of motion. They showed
that the temperature of a body can be raised by the expendi-
ture of mechanical energy. Somewhat later, therefore, when
Robert Mayer began to perceive the equivalence of the sev-
eral forms of energy he had at hand an established con-
nection between mechanical and thermal phenomena so he
included heat as a form of energy to account for the observed
changes in temperature and, equally important, to provide
something which could be conserved. And this last feature
provides the essence of our present notion of heat energy; it
is something we postulate as existing in order to make the
law of conservation of energy valid. As Bridgman states it:
"The heat concept is in the general case a sort of wastebasket
concept, defined negatively in terms of the energy left over
when all other forms are allowed for."

Even today, however, with the caloric theory thoroughly
discredited, vestiges of it, chiefly certain habits of thinking,
persist, and, since they have a definite usefulness, they are
likely to continue to survive. The most obvious instance is
the term "flow of heat", still in common use, as witness the
title of these lectures, which implies a flow similar to that
of a liquid. This concept is useful, indeed the quantitative
statement of the dynamic theory of heat by Fourier was
based on precisely this sort of fluid flow, and it does no harm
so long as it is realized that no physical significance can be
directly given to heat and that there are no operations for
measuring it directly. All we measure is temperature distri-
bution and rate of change of temperature.

Another remnant of the caloric theory is the idea of heat

capacity, which implies the capacity of a body to hold the caloric fluid. Still another is the view that heat is indestructible and uncreatable which persists, in modified form, in the Law of Conservation of Energy, which is also called the First Law of Thermodynamics.

Finally a fourth example, and an exceedingly significant one, is the concept of temperature as a level or potential, for this is the essence of the Second Law of Thermodynamics, which states that heat does not of itself flow—note the idea of flow again—from a place of low temperature to one of higher temperature. Once heat became classified as one of the several forms of energy the retention, indeed, the elaboration, of this feature of the caloric theory was inevitable for the concept of potential is essential in dealing with energy. Thus, the complete characterization of any given amount of energy requires the specification of two factors: A quantity factor, which indicates the amount of energy, and a quality, or potential, factor which indicates the quality of the energy, or more specifically, the potential or level at which it is available. In dealing with the transfer of energy, as in the case of the transfer of matter, this potential factor is particularly significant because it determines not only the direction of flow but, together with the resistance to flow, which is characteristic of the particular system considered, determines the net rate of transfer. For example, in hydrodynamics, the rate of flow of water is determined by the frictional resistance of the pipe and the difference in level through which the water can fall; in electrodynamics, the rate of flow of electricity, that is, the current strength in amperes, is determined by the resistance of the circuit and the potential drop or voltage, and is given numerically by the ratio of voltage to electrical resistance; in thermodynamics, the rate of heat flow is determined by the thermal resistance of the body through which heat is flowing and by the drop in thermal potential, that is,

Table I
Thermal Conductivity of Various Materials at Room Temperature Expressed in cal.cm^{-1}.sec^{-1}.°C^{-1}

Air	0.000057
Hydrogen	0.000133
Alcohol	0.00046
Water	0.00150
Glass	0.0020
Fused Quartz	0.003
Ice	0.0050
Rock Salt	0.0167
Quartz	0.018
Alumina	0.025
Iron	0.150
Aluminum	0.480
Silver	1.00

by the difference in temperature, and is numerically equal to the ratio of this temperature gradient to the thermal resistance. The correspondence between the potential and resistance factors for electrical energy and those for thermal energy is a very precise and useful one, as there will be frequent occasion to point out in the course of these lectures. One outstanding application of this analogy is the solution of problems in heat flow, which are too complex or difficult to be treated directly, by a study of the behavior of an electrical circuit which is the analogue of the thermal circuit in question, a method which has been successfully employed by Paschkis and others.[1]

NATURE OF METALLIC CONDUCTION

' Since heat is a manifestation of molecular motion and conduction of heat is basically the transmission of heat energy from molecule to molecule it is natural to wonder why the thermal conductivity of materials differs so widely. For example, the data in Table I show that the conductivity of silver is more than a thousand times greater than that of air and fifty times greater than that of rock salt. What is it, therefore, that makes a solid conduct so much better than a gas or a liquid? Why does a crystal such as quartz conduct better

[1]See, for example, the article by Swain in *Power*, July, 1941.

than an amorphous solid such as vitreous silica, and why is
a metal so much better a conductor than a nonmetal? A com-
plete answer to these questions cannot be given in our pres-
ent state of knowledge but a fairly satisfactory answer is
found partly in the special properties of the crystal lattice
and partly in the difference between the electronic structure
of metals and that of nonmetals.

Conduction in a gas, liquid, or amorphous solid, is due
almost entirely to heat transfer from molecule to molecule,
a process which may be thought of as a kind of diffusion of
energy. Such a process is inherently relatively slow. On the
other hand, in a crystal there is in addition a transfer of heat
through a vibratory motion of the crystal lattice as a whole.
Thus, a crystal may be pictured as a system of coupled reso-
nators in which long wave oscillations are superposed on
molecular motion. These so-called thermoelastic waves, which
develop properly only in a crystal, have a wave-length some-
what greater than the distance between lattice points, are not
purely harmonic, and may be disturbed by the heat flow itself;
they are also scattered or distorted by imperfections in the
lattice and are thereby lessened in intensity. These waves
are not imaginary for evidence of their presence has been
found by Preston in the X-ray diffraction pattern of crystals
at high temperature, as is illustrated in Fig. 1 by the diffrac-
tion pattern obtained from a crystal of aluminum at room
temperature and at 500 degrees Cent. (930 degrees Fahr.).
The hexagonal cluster of spots around the center of the pat-
tern, which is faintly visible at room temperature but is
intense at 500 degrees Cent., is due to thermoelastic vibra-
tions of the lattice. It is the contribution of such lattice con-
duction which makes a crystal a much better conductor of
heat than a gas, liquid, or amorphous solid.

Yet lattice conduction does not suffice to account for the
relatively enormous conductivity of a metal, which can be

explained only by going a step beyond the structure of the crystal to the electronic structure of the metal atom itself. Of the electrons which surround the nucleus of each atom, the innermost ones are held so tightly that it is virtually impossible to remove them, but the outermost electrons, that is, the valence electrons, can be removed much more readily.
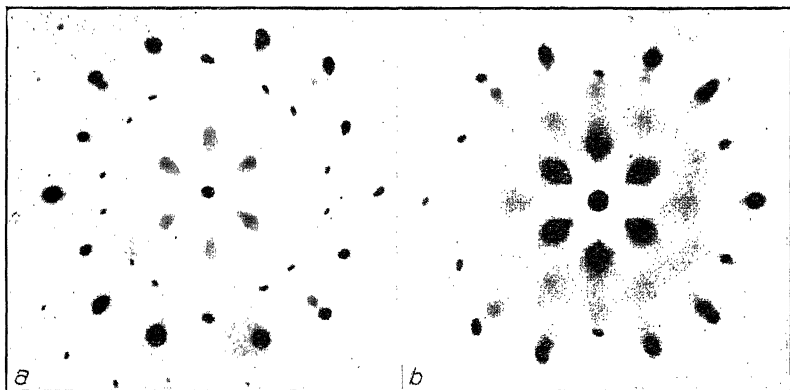
Fig. 1—X-ray Diffraction Pattern of Aluminum Showing Spots Due to Thermo-elastic Waves. a, Room Temperature; b, 500 Degrees Cent. (930 Degrees Fahr.). (After Preston.)

In a crystal of a nonmetallic compound, these valence electrons are transferred from one atom to another or are shared between atoms, to form the chemical bond which ties the several atoms into a molecule; but in a metal crystal, in which substantially all the atoms are alike, these valence electrons simply become detached and are free to move within the space between the positive ions which constitute the remainder of the atom.

A typical metal crystal may therefore be regarded as an orderly array óf positive ions, each ion being an atom of the metal with its valence electrons removed, surrounded by a cloud of electrons which are not attached to any particular atom but are free to move at random through the metal. This structure is thus equivalent to a network of positive ions

immersed in a gas composed of free electrons, as is illustrated
schematically in Fig. 2.

When the temperature and electric potential are uniform
throughout the metal, the electrons move entirely at random,
colliding occasionally with each other, but much more fre-
quently with the ions forming the lattice, and being thereby
scattered or deflected so that on the whole there is no resultant
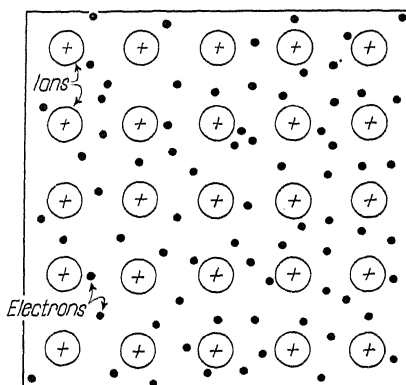
Fig. 2a—Schematic Diagram of the
Structure of a Metal Showing Positive Ions
Surrounded by an Electron "Gas."

migration of electrons in any given direction. On the other
hand, when there is a difference in temperature or electric
potential within the metal, there is a general drift of elec-
trons in the direction of the temperature or potential drop
which results in a flow of energy, and it is this drift of the
free electrons which is responsible for the high thermal and
electrical conductivity of a metal as compared to that of a
nonmetallic material. Moreover, the interruption of this drift,
arising from collisions between the electrons and the ions of
the lattice, constitutes the thermal and electric resistance of
the metal, for the effect of the drift is destroyed for any
given electron by a collision.

The kinetic energy of the free electrons may be taken as
that of a molecule of an ideal gas at the same temperature,

but since the mass of the electron is very much less than that of even the lightest gas molecule, hydrogen, the velocity of thermal agitation is very great, being about 10,000,000 centimeters per second, in fact. Now when a voltage drop is imposed on the metal, there is, as we have seen, a resultant drift of the electrons. It must be emphasized that this drift is only a general trend and is not necessarily apparent in the motion of every electron. Some electrons will still be moving one way, some another, but taking a statistical average of the displacement of all the electrons there is a net migration which is most conveniently regarded as the effective acquisition by each electron of a uniform velocity in the direction of the voltage drop. For any ordinary current density, the velocity of drift is small compared to that of thermal agitation. For example, in copper at a current density of 10,000 amperes per square centimeter the velocity of drift is only one-hundredth of the normal gas velocity.

In the corresponding thermal case, when a piece of metal is heated at one end, the electrons tend to drift from the hotter to the colder parts, just as do the molecules of a gas whose temperature is not uniform, but this drift is again interfered with by collisions between the electrons and the ions forming the lattice. It should be noted that although the velocity of this drift may be appreciable, the actual distance drifted by any group of electrons is small, being, in fact, of atomic dimensions. That is, there is no large scale migration of electrons over a macroscopic distance. Were such a migration to occur the electrons would drain from one part of the metal to another and there would be very large electrical effects accompanying heat flow and vice versa. As a matter of fact, there are electrical effects associated with the flow of heat in a metal in that there is a difference in electrical potential between the hot and cold ends of a piece of metal whose temperature is not uniform, but it is relatively small and

indicates that the actual displacement of the conduction electrons is slight.

To recapitulate, the mechanism of conduction of heat or electricity in metals can be represented by the following picture, which, though greatly oversimplified and not entirely consistent, is nevertheless a useful working view. When the temperature and electrical potential are uniform throughout a piece of metal, the free electrons move at random within the framework of the crystal lattice but are deflected and scattered by collisions with other electrons or with ions in the lattice. If, however, an electromotive force is impressed on the metal, these electrons acquire a drift in the direction of the potential drop which carries some of them out of the metal at one end and leaves a few vacancies at the other, these vacancies being then filled by migration of electrons from the conductor with which that end of the metal is in contact. There is thus a passage of electricity with only a relatively small displacement of the electrons. Obviously, such a process needs a closed circuit in order that no part of the system be drained of electrons while other portions acquire a surplus.

If, now, instead of having an electromotive force impressed on the metal, the piece is heated at one end, the situation is somewhat different, for the significant transfer in this case is not so much the electrons themselves as their motion. At the high temperature end of the metal, the electrons acquire a velocity of thermal agitation higher than the average throughout the metal, together with a tendency to drift toward the cooler part. The increased energy of vibration and the tendency to drift is in this way communicated to other electrons which in turn pass them on to still others. Heat energy is thus transferred through the metal fairly rapidly with only a very small net displacement of electrons. It will be noted that this process which transfers electronic

motion rather than electrons does not require a closed circuit as in the transfer of electricity.

Since the characteristically high thermal conductivity of a metal results from the presence of conduction electrons, we are now faced with a further question. Why does the conductivity of different metals vary so widely? Why, for instance, is the conductivity of pure silver six times that of iron which in turn has a conductivity three times that of stainless steel? This question carries us very close to the limits of present knowledge and a complete answer is not as yet available but there is again a partial answer which, though incomplete and greatly oversimplified, is nevertheless useful. It involves the properties of these conduction electrons.

On the basis of the electron gas model of the structure of a metal, the thermal, or electrical, conductivity is influenced: (a) to some extent by the number of electrons present per unit volume, which implies that conductivity varies with sub-atomic structure; (b) by the velocity of the conduction electrons; (c) to much the greatest extent by the way in which collisions with other electrons or with ions in the lattice interfere with the general drift of the electrons under the gradient in temperature or electrical potential. Since the effective cross section of an ion in the lattice is about 100,000 times greater than that of an electron, it is clear that the number of collisions between two electrons is negligible compared to the number of collisions between an electron and an ion, so that our chief interest lies in the interaction between the electrons and the lattice. This interaction is specifically one between the electric fields surrounding the two types of particle but it is permissible and convenient to regard this interaction as a mechanical collision. As a consequence of this interaction, the number of collisions is very small if the lattice is perfect and is increased by anything which disturbs or distorts the periodicity.

In further studying the electron gas model, it is convenient to think of the motion of the electrons as being characterized by a mean free path, analogous to the mean free path of the molecules of a gas, which is defined as the length of path travelled in a straight line between collisions or deflections. Obviously, this is not the same for all electrons, but under a given set of conditions, a properly computed average can be used to characterize the behavior of the electrons, just as temperature is characterized by a proper statistical average of molecular velocities. It should be noted that this electronic mean free path, like that of a gas molecule, is of atomic dimensions, and that it is influenced by temperature.

On the basis of this concept it is possible to evaluate that part of thermal conduction which is electronic in terms of the mean free path of the electrons. The rigorous and quantitative derivation of this relation involves the use of quantum mechanics and is therefore a matter for the specialist, but the essential features of the result can be obtained more easily by the following simple, if not too rigorous, reasoning.

Imagine a small unit cube of atomic dimensions within a larger body through which heat is flowing, such as is shown in Fig. 2b. This cube is chosen so that there are two opposite faces perpendicular to the direction of flow, over each of which the thermal potential, that is, the temperature, is constant and uniform. The strength of the thermal current passing through the cube, that is, the number of calories flowing per second, a quantity we shall designate by q, depends: (a) upon the difference in temperature $(t_1 - t_2 = \Delta t)$ between the two isothermal faces; (b) on the number of electrons per second, n, passing completely through the cube; and (c) on the amount of thermal energy (Θ) associated with each electron. The rate of flow per unit difference in temperature may therefore be expressed as:

$$\frac{q}{\Delta t} \text{ is proportional to n and to } \Theta \qquad (1a)$$

But the ratio q/Δt is a direct measure of the ability of the body to conduct heat, that is, it is the thermal conductivity of the metal of which the cube is made, and it is represented by the symbol K. Moreover, n, the number of electrons traversing the cube per second, is proportional to the mean free path, l, since the longer this path, the smaller is the fraction of the
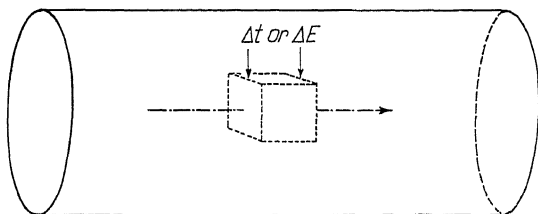
$$\frac{q}{\Delta t} \text{ is proportional to } n \text{ and } \theta$$
$$K = alT$$
$$\frac{l}{\Delta E} \text{ is proportional to } n \text{ and } e$$
$$\sigma = b'le = bl$$
$$\frac{K}{\sigma T} = \frac{a}{b} = A$$

Fig. 2b—Illustration of Unit Cube Used in Deriving Equations 1b and 2b.

number of electrons entering the cube which is scattered or deflected in passage and the greater is the number of electrons which get through.

Finally, since the thermal energy, Θ, associated with each electron is kinetic energy, its amount is proportional to the absolute temperature, T, for the absolute temperature is, in essence, simply a convenient criterion of the mean thermal energy of the particles in question. Combining all these relationships by means of a single proportionality constant, equation (1a) becomes

$$K = a \, l \, T \tag{1b}$$

Where K is the thermal conductivity of the metal, l the mean free path of the electrons, T the mean temperature on the absolute scale and "a" the proportionality constant. Since, as

is shown below, the mean free path varies inversely with the amplitude of the thermal vibration of the ions in the lattice, which is again a function of the temperature, it is evident that the thermal conductivity is doubly influenced by temperature; first, through the amount of thermal energy associated with each electron, an influence which is represented explicitly in equation (1b) by the term T; second, through the effect of temperature on the lattice, and therefore on the mean free path, which is represented implicitly in equation (1b) by the term l. These two influences in general oppose each other since the thermal energy increases with temperature whereas the mean free path decreases with temperature because of increased interaction with the lattice, hence the net variation in K is the resultant of the variation of these two factors. If the change in mean free path with temperature is large compared to the change in the value of the temperature itself, the thermal conductivity decreases; if the change is small compared to the change in the value of T, the conductivity increases with temperature.

It is clear from this equation that the longer the mean free path the higher is the thermal conductivity of the metal, and since anything which in any way distorts the periodicity of the lattice lessens the mean free path, any disturbance in the lattice results directly in a lower thermal conductivity. If it were possible to obtain a perfect crystal composed of but one kind of atom and entirely free from thermal vibration, the mean free path would be increased to an extent that the thermal resistance of the crystal would tend to vanish. But no such lattice exists because many factors prevent its realization. There is first the presence of foreign atoms which distort the lattice and destroy its periodicity, as shown schematically in Fig. 3. On this basis, the conductivity of a pure metal should be, and is, greater than that of an impure sample of the same metal, as illustrated by the fact that the con-

ductivity of iron is three times greater than that of stainless steel. Moreover, the disturbance created by the addition of the first few foreign atoms in a virtually perfect lattice, such as that of a pure metal, might be expected to produce a greater decrease in conductivity than the addition of the same number of foreign atoms to an alloy which already has a dis-
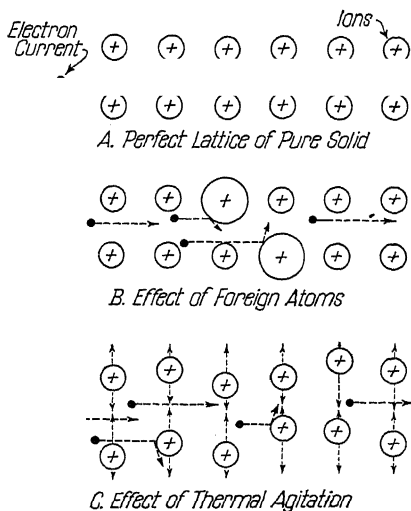
Fig. 3—Schematic Illustration of the Effect of Lattice Disturbance on the Electron Current.

turbed lattice. Likewise any strain in the lattice produces distortion and so results in a lessened conductivity. Finally, the thermal vibrations of the atoms themselves effectively distort the lattice, as is illustrated rather crudely in Fig. 3, and cause a scattering of electrons with a resultant decrease in conductivity. In a pure metal most of the thermal or electrical resistance arises from such thermal agitation. Since at a given temperature the amplitude of the thermal agitation differs for different metals because of the difference in atomic structure, the mean free path of the electrons, and, in consequence, the thermal conductivity, varies from metal to metal.

RELATION BETWEEN THERMAL AND ELECTRICAL CONDUCTIVITY OF A METAL

Since both the thermal and electrical conductivity of a metal depend upon the presence of free electrons, there should be some relatively simple relation between them, and such a relation has, in fact, been found.

Thus it can be shown by reasoning similar to that used in deriving equation (1b) that electrical conductivity is also related to the mean free path. Returning to the unit cube shown in Fig. 2b, suppose that instead of a thermal current an electrical current is flowing and that the opposite faces of the cube are at constant and uniform electrical potential, that is, that there is a constant voltage drop, ΔE, through the cube. The strength of the electric current, that is, the rate of transfer of electricity through the cube, expressed as amperes or coulombs per second, and designated by I, depends upon the voltage drop, ΔE, upon the number of electrons, n, passing through the cube per second and upon the amount of electrical energy (e) associated with each electron; hence, the rate of flow for a unit potential drop can be expressed as:

$$I/\Delta E \text{ is proportional to n and to e} \qquad (2a)$$

Now $I/\Delta E$ is the electrical conductivity of the metal, which we shall represent by the symbol σ, and n is, as before, proportional to the mean free path of the electrons. The charge associated with each electron, e, is a constant and in that respect differs from the amount of thermal energy associated with each electron, which increases with temperature. Expressing these relationships by means of a single proportionality constant

$$\sigma = b' \, l \, e \qquad (2b)$$

where σ is the electrical conductivity, l is the mean free path of the electrons, and b' is a constant. This relation is pre-

cisely analogous to equation (1b), the term e representing the electrical energy being the counterpart of the term T in equation (1b) which represents the thermal energy associated with an electron. Since the value of e is constant, it may be included with b' to give the simpler relation

$$\sigma = b \, 1 \tag{2c}$$

The factors which comprise the constants a and b in equation (1b) and (2c) respectively are all fundamental constants which can be evaluated by means of quantum mechanics, and are described in the works of Seitz and of Hume-Rothery cited in the bibliography at the end of this chapter.

It should be noted that although a temperature term does not appear explicitly in equations (2b) and (2c) there is still, as in the case of thermal conductivity, an indirect influence of temperature through its effect on the mean free path. Since the mean free path decreases with temperature because of the increased thermal vibration of the lattice, the electrical conductivity likewise decreases with temperature.

It should also be pointed out that although equations (2b) and (2c) for electrical conductivity are not formally identical with equation (1b) for thermal conductivity, the difference is more apparent than real, because it results chiefly from the choice of different units for expressing the energy. If, for example, all the energy terms were expressed in ergs, the equations would be substantially alike. Moreover, if the thermal and electrical conductivity are expressed in terms of the operations of measurement, they are precisely analogous, because each quantity is, in essence, a measure of the rate at which energy is transported between two opposite faces of a unit cube of material, when there is a unit potential gradient between these faces.

It is at once evident that equations (1b) and (2c) can be

Table II

Values of the Wiedemann-Franz Ratio and of the Lorentz Constant for Different Metals at Different Temperatures

Metal	Wiedemann-Franz Ratio K/σ Degrees Cent.		Lorentz Number $L = K/\sigma T$ Degrees Cent	
	0	100	0	100
Aluminum	136×10^{-8}	199×10^{-8}	.....	0.533×10^{-8}
Bismuth	216	257	0.791×10^{-8}	0.690
Cadmium	158	216	0.578	0.580
Copper	145	207	0.533	0.556
Gold	153	214	0.561	0.573
Iridium	163	222	0.595	0.594
Iron	162	...	0.590	.....
Lead	162	228	0.590	0.612
Magnesium	153	206	.....	0.552
Molybdenum	170	248	0.623	0.666
Nickel	...	206	.....	0.545
Palladium	169	244	0.619	0.654
Platinum	164	232	0.600	0.621
Rhodium	167	226	0.614	0.606
Silver	150	216	0.552	0.566
Tin	164	222	0.600	0.595
Tungsten	198	285	0.727	0.764
Zinc	150	207	0.552	0.556

Ideal value of L = 0.536×10^{-8}

combined to give a direct relation between the electrical and thermal conductivity of a metal; thus, dividing equation (1b) by equation (2c)

$$\frac{K}{\sigma T} : \frac{a}{b} \quad A \qquad (3)$$

where A is a new constant, whose value again depends somewhat on the assumptions made, but which involves only fundamental physical constants and should, therefore, be the same for all metals. This relation proves, when checked experimentally, to be approximately valid and is useful in correlating values of electrical and thermal conductivity.

The fact that the ratio of thermal conductivity to electrical conductivity is approximately constant at room temperature was first discovered empirically in 1853 by Wiedemann and Franz, and this ratio is therefore called the Wiedemann-Franz ratio. Although the value of this ratio is sensibly constant for most pure metals at a given temperature, it varies considerably with temperature, as is evident from the data given in Table II. In general, the value of this ratio

K/σ increases with increasing temperature because at all temperatures above about 200 degrees absolute the electrical conductivity, σ, decreases much more rapidly with temperature than does the thermal conductivity.

Many years after the discovery of this relation, Lorentz, on the basis of the free-electron theory, derived a relation of the same form as equation (3), and showed that by adding a temperature factor to the Wiedemann-Franz ratio, the value of K/σT should be a universal constant, namely, 0.536×10^{-8}, when K is expressed in cal. sec^{-1} . cm^{-1} . °C^{-1}, and σ is expressed in microhm^{-1} . cm^{-1}. This ratio is substantially constant for most pure metals at room temperature and above, as is evident from the data in Table II; moreover, the experimentally determined values of the ratio come surprisingly close to the value calculated by Lorentz. The ratio K/σT is often called the Wiedemann-Franz ratio, but should more properly be called the Lorentz constant.

When applied to alloys, both the Wiedemann-Franz ratio and the Lorentz constant show a greater variation than for pure metals, and in most cases the ratio increases slightly as the proportion of alloying element increases.

As more refined measurements of thermal conductivity have become available it has been found that a closer correlation between electrical conductivity and thermal conductivity is obtained by means of an equation of the form

$$K = A \, \sigma \, T + B \qquad (4)$$

where A and B are constants. In this equation the term AσT represents the contribution of electronic conduction whereas B represents the contribution of lattice conduction to the total transmission of heat. Lattice conduction is a factor only in the passage of heat, and not in the passage of the electric current, hence thermal resistance does not vanish as electrical resistance becomes zero. The constant B gives, in effect, the

Table III
Equations of Form $K = A \sigma T + B$
Representing Relation Between Thermal and Electrical Conductivity

When K is expressed in $cal.cm^{-1}.sec^{-1}.°C^{-1}$
and
σ in $microhm^{-1}.cm^{-1}.$

Iron Alloys	$K = 0.625 \times 10^{-8} \sigma T + 0.006$
Copper Alloys	$K = 0.571 \times 10^{-8} \sigma T + 0.018$
Aluminum Alloys	$K = 0.502 \times 10^{-8} \sigma T + 0.003$
Magnesium Alloys	$K = 0.526 \times 10^{-8} \sigma T + 0.027$

Value of Lorentz Constant $= 0.536 \times 10^{-8}$

thermal conductivity of the material considered as a non-metallic crystalline solid. Numerous tests on alloys of iron, of copper, of aluminum and of magnesium have shown that the data for these systems can be represented satisfactorily within about 10 per cent by the equations shown in Table III.

Fig. 4—Thermal Conductivity of Ferrous Alloys Plotted Against the Product of Electrical Conductivity and Absolute Temperature (After Powell.)

The extent to which the equations fit the observations for a particular group of alloys is shown in Figs. 4, 5, 6, and 7, in which the thermal conductivity is plotted against the product of electrical conductivity and absolute temperature. The slope of the lines in these Figures gives the constant A, the intercept on the vertical axis, the constant B.

It will be noted that in these equations the value of the constant A in no case differs markedly from that of the Lorentz constant, namely, 0.536×10^{-8}. Moreover, the value of B is of the same order of magnitude as the thermal conductivity of pure nonmetallic crystals, and is generally greater
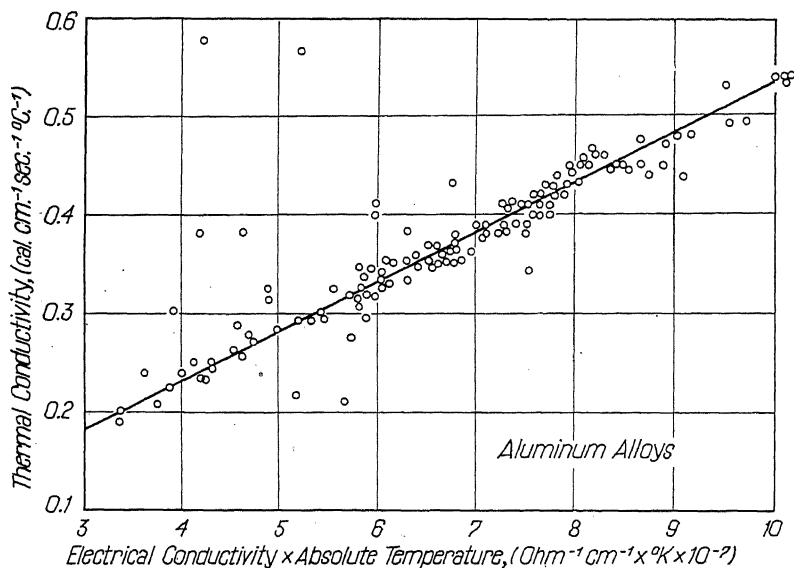
Fig. 5—Thermal Conductivity of Aluminum Alloys Plotted Against the Product of Electrical Conductivity and Absolute Temperature. (After Kempf, Smith and Taylor.)

than the conductivity of an amorphous solid, such as glass, as is evident by comparison with the data given in Table I.

These equations are useful in many ways. They show first the high degree of correspondence between variations in electrical and thermal conductivity. A variation in thermal conductivity can therefore be predicted from data on electrical conductivity even though no direct thermal measurements are available. Again, if high accuracy is not needed—and it generally is not needed in most calculations of heat flow—a fairly reliable estimate of the difficultly measurable thermal

conductivity can be made from data on the readily determin-
able electrical conductivity. For electrical measurements are
carried out very much more easily and with much greater
precision than the corresponding thermal measurements. In
electrical conduction, using ordinary voltages and current
strengths, a metal is almost completely insulated by the sur-
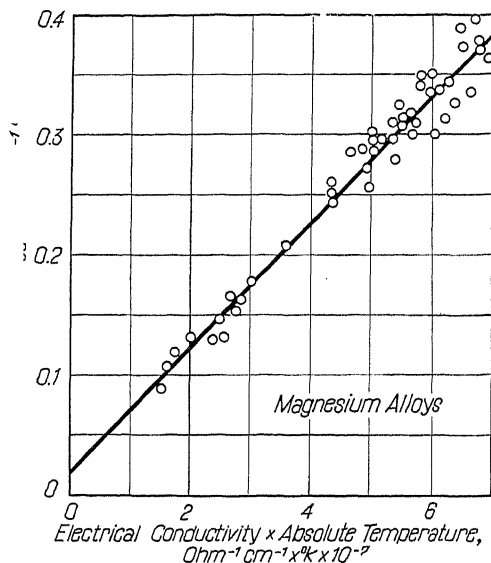
Fig. 6—Thermal Conductivity of Magnesium Al-
loys Plotted Against the Product of Electrical Con-
ductivity and Absolute Temperature. (After Powell.)

rounding medium, such as air, oil, or rubber, and the whole of
the electricity passing can be closely controlled and measured.
On the other hand, in the case of thermal conduction, there
is no perfect thermal insulator, hence, some heat is continually
being transferred to or from the surrounding medium, so that
any absolute determination of the flow of heat in the metal
is made very difficult, and even a comparative measurement
is subject to appreciable error.

    To sum up, the thermal current in metals is carried to a

limited extent by vibrations of the crystal lattice, the so-called thermoelastic waves, but to a very much greater extent by free electrons moving within the metal, and it is this electronic conduction which makes metals enormously better conductors than nonmetals. These free electrons behave as if they were a gas surrounding the lattice of positive ions which
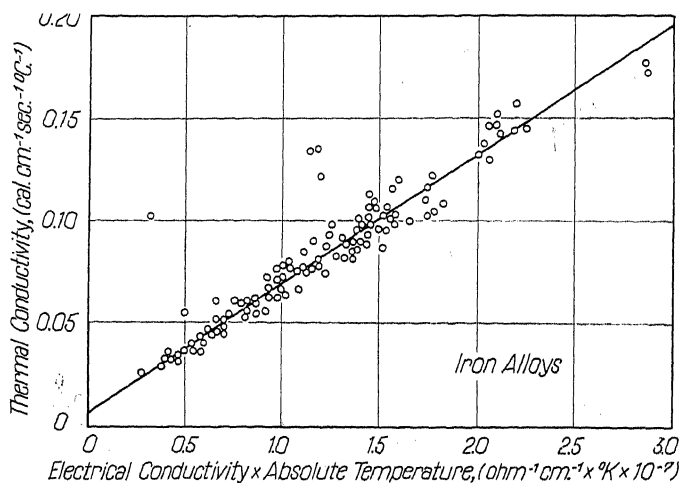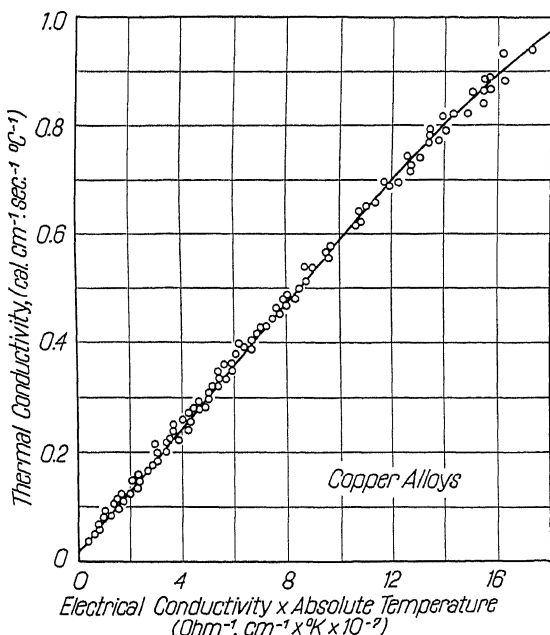

Fig. 7—Thermal Conductivity of Copper Alloys Plotted Against the Product of Electrical Conductivity and Absolute Temperature. (After Smith and Palmer.)

forms the framework of the metal and it is useful to think of their motion in terms of a mean free path, analogous to the mean free path of the molecules of a gas, which is defined as the distance they travel in a straight line without being reflected by a collision or scattered by an imperfection in the periodic electric field set up by the ions in the lattice. The longer the mean free path of the free electrons, the greater is the thermal conductivity of the metal. This picture is, of

course, considerably over-simplified but it is useful because, quite apart from its exactness, it provides a consistent pattern for substantially all the experimental data on the conduction of heat in metals, as is evident from the discussion in the next chapter.

On the basis of this concept of free electrons, the ratio of the thermal conductivity to the product of the electrical conductivity and the absolute temperature should have a constant value of 0.536×10^{-8} for all metals. Experimental check shows that this ratio is approximately constant. When the contribution of lattice conduction is also taken into account, the thermal conductivity of metals (K) can be expressed by the relation

$$K = A \sigma T + B$$

where σ is the electrical conductivity, T is the absolute temperature, and A and B are constants, A being approximately 0.536×10^{-8} as above and B having a value of the order of magnitude of the thermal conductivity of a nonmetallic crystal. For a given metal and its alloys this relation holds with fair accuracy.

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Chapter II

FACTORS INFLUENCING THE THERMAL
CONDUCTIVITY OF METALS

The first chapter was devoted to a discussion of the prop-
erties of the free or conduction electrons which provide a
basis for explaining why a metal is a better conductor of heat
and electricity than a nonmetal and why all metals do not con-
duct equally well. The conclusions drawn from this discussion
were, however, chiefly qualitative. Although they suggest the
nature of the influence of certain factors, they by no means
cover all possible variations, and in no case do they indicate
the magnitude of any variation. For a knowledge of how the
thermal conductivity varies, that is, to obtain quantitative
data on a specific variation, it is necessary to go directly to
experiment, and in this chapter we shall review the experi-
mental data bearing on the effect of different factors. To do
this systematically there are listed in Table IV all the factors

Table IV
Factors Which May Influence the Thermal Conductivity of a Metal

A. Atomic Structure of Pure Metal
B. Chemical Composition
    1. Influence of impurities in solid solution.
    2. Presence of two or more phases.
C. Physical Constitution
    1. Phase changes.
    2. Energy changes such as loss of ferromagnetism.
    3. Directional effects.
    4. Grain size.
    5. Porosity—metal powders and sintered products.
D. Temperature
E. Pressure and Deformation

which seem likely to have a significant influence, and we shall
evaluate, in so far as the available data permit, the effect of
a change in each variable when all the others are held sensibly
constant.

In examining the experimental data in this way it must
again be emphasized that the precise determination of ther-

mal conductivity is very difficult, chiefly because there is available no perfect thermal insulator to confine the thermal current to the path desired; hence, the measurements reported in the literature are almost invariably subject to considerable error and occasionally show an irregular variation which tends to confuse interpretation.

ATOMIC STRUCTURE OF PURE METALS

Since metallic conduction is associated with the presence of free electrons, there ought logically to be a relation between the thermal conductivity of a metallic element and its atomic structure, which is most easily classified by its position in the Periodic System, or by what is equivalent, its atomic number. Such a relation is, in fact, observed when comparison is made on the proper basis, but this basis differs from that commonly used. When thermal conductivity expressed in the ordinary units is plotted against atomic weight or atomic number, no obvious regularity appears because the common unit for thermal conductivity is based on unit volume and a unit volume of different metals contains a different number of atoms and is therefore not a proper basis for studying atomic effects. Moreover, at any given temperature, the effect of thermal agitation varies appreciably from metal to metal, and such agitation has, as was pointed out in the first chapter, a direct influence on the thermal conductivity. If, on the other hand, one uses the atomic thermal conductivity, that is, the effective conductivity per atom, rather than the conductivity per unit volume, and if, in addition, a suitable correction is made for the difference in thermal agitation, then the expected regularities appear. Unfortunately, the necessary data for making this correction are lacking except in a few cases, so that the variation of atomic thermal conductivity with atomic number, though fairly evident, does not serve well as an

illustration because of this paucity of data. But adequate data for the corresponding variation in electrical conductivity are available, and serve to show more fully what the variation is (see Fig. 8). The few data for thermal conductivity are entirely in accord with the data for electrical conductivity so that conclusions drawn from Fig. 8 can be safely applied to thermal conductivity as well.
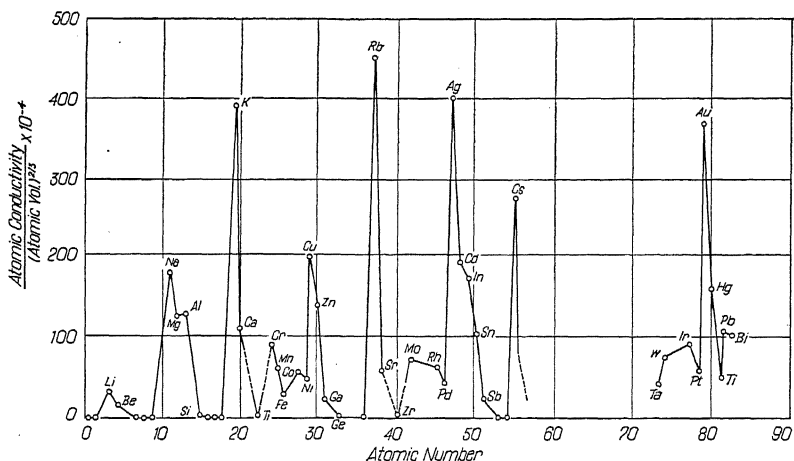
Fig. 8—The Variation of the Atomic Electrical Conductivity of the Metallic Elements with Atomic Number. (After Hume-Rothery.)

The most evident regularity is that the atomic conductivity is much the greatest for the univalent elements of both long and short periods of the Periodic System; thus, the best conductors are the alkali metals and copper, silver, and gold. These are the elements whose structure consists of a complete shell of inner electrons plus one valence electron. It is also evident that there is a marked decrease in conductivity in passing from these univalent elements to the divalent elements which immediately follow them, for example, in passing from sodium to magnesium or from copper to zinc. Moreover, the conductivity of those elements, such as silicon, which possess some, but not all, of the characteristics of a metal, and of the

metals of complex electronic structure, such as zirconium or titanium, is low. A good many more conclusions as to the detailed atomic energy levels can be drawn but these are of interest mainly to students of the structure of solids and need not be discussed here.

INFLUENCE OF COMPOSITION AND CONSTITUTION

Effect of Impurity—The simplest variation of composition is the addition of a relatively small amount of impurity to a pure metal. In accord with expectation, based upon the disturbing effect of foreign atoms upon the lattice (Fig. 3,
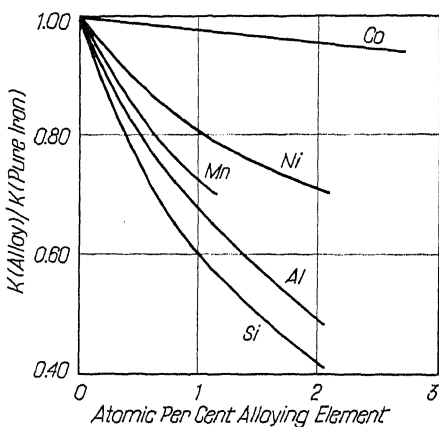
Fig. 9—The Influence of Impurity on the Thermal Conductivity of Iron. (After Sedstrom.)

Chapter I, page 17), the presence of foreign atoms results in a marked decrease in thermal conductivity, as is shown in Fig. 9, in which the thermal conductivity of a number of dilute iron alloys, expressed as the ratio of the conductivity of the alloy to that of pure iron, is plotted against atomic per cent of alloying element. In comparisons of this sort, atomic per cent is the proper basis since it shows the variation in terms of the number of foreign atoms present instead

of the weight of foreign atoms present, as is the case when weight per cent is used. In addition to demonstrating the effect of the presence of foreign atoms, the curves in Fig. 9 illustrate two other significant points. First, that the addition of the first few atoms of impurity is far more effective in lessening the thermal conductivity than is the addition of an equal number of atoms of the same impurity when an appreciable amount of impurity is already present. For example, starting with 100 atoms of iron, the replacement of one atom of iron by an atom of nickel lessens the conductivity of the metal by about 20 per cent, whereas the replacement of a second atom results in an additional decrease of only 10 per cent. Second, that an impurity whose atomic structure resembles that of the solvent metal produces a smaller change in conductivity than an impurity whose structure differs greatly from that of the solvent metal; in this respect the metalloids, such as carbon and silicon, are notably effective in lessening the conductivity of a metal. Thus, in Fig. 9, the metals nickel and cobalt, which have a structure similar to that of iron, cause the least decrease in conductivity, whereas aluminum, which has quite a different structure, produces a much greater change and finally, silicon, a metalloid, produces the greatest decrease of any of the alloying elements shown.

The curves of Fig. 9 are based on the presence of but one alloying element; when two or more are present, their combined effect on the resistance of the metal, that is, the reciprocal of the conductivity, is roughly additive, provided the impurities are present in relatively small amount, that is, if the solid solution can be considered dilute. A number of attempts have been made, particularly for steels, to set up a relation by means of which the thermal conductivity or thermal resistance of the alloy can be calculated from its composition, but they have in general met with distinctly limited success. One of the most satisfactory is the equation derived by

Masumoto for the thermal resistance of steels, which is:

$$1/K = 5.74 + 2.43 \ (\% \ C) + 5.08 \ (\% \ Si) + 2.46 \ (\% \ Mn)$$

In order to show to what extent this relation reproduces the experimental data, calculated values for a number of steels have been plotted in Fig. 10 against the directly determined value. If the correspondence were exact the plotted points
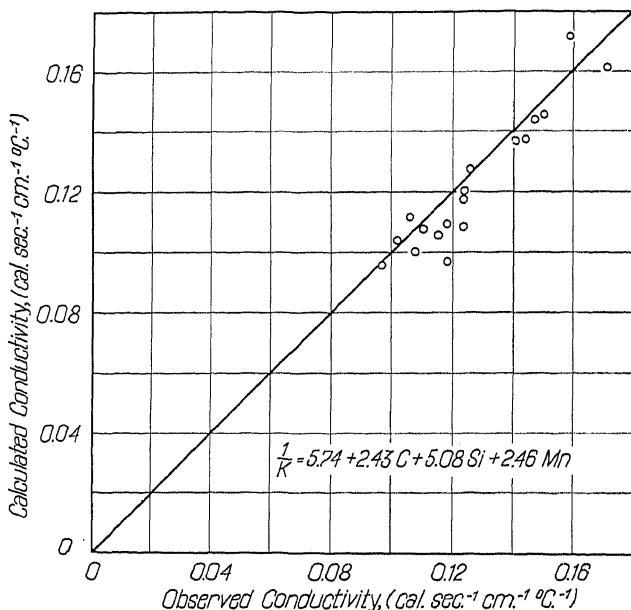
Fig. 10—Comparison of the Thermal Conductivity of Steel as Calculated by Means of Masumoto's Equation with that Directly Determined.

should all fall directly upon the heavy black line, whereas if the calculated value is too high or too low, the point falls above or below the line respectively. It is evident from Fig. 10 that this relation fits the observations reasonably well, but has a tendency to give results which are slightly low.

Another significant point to be noted about the effect of the presence of impurity is that the higher the conductivity of the solvent metal, the greater is the decrease in conductivity

caused by the presence of a given number of foreign atoms. This is illustrated in Fig. 11 by data on the effect of nickel on the conductivity of copper and iron, the former having a conductivity some six times that of the latter. The data, plotted for ease of comparison in the form of the ratio of the
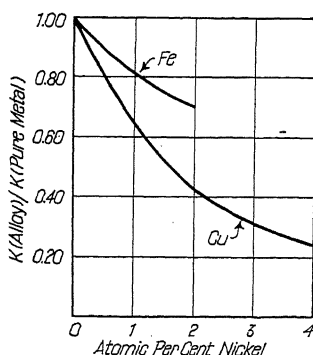
Fig. 11—The Relative Influence of a Given Metal (Nickel) on the Thermal Conductivity of a Metal Which is a Good Conductor (Copper) and That of a Poorer Conductor (Iron).

conductivity of the alloy to that of the pure metal, show that when nickel is present to the extent of 2 atoms per hundred, the conductivity of copper, the better conductor, is lessened by some 60 per cent, whereas, that of iron, the poorer conductor, is lessened but 15 per cent.

Alloys with Continuous Solid Solution—From the dilute solid solutions just discussed it is but a simple step to alloy systems in which there is a continuous series of homogeneous solid solutions. The variation of conductivity in such alloy systems is essentially the same as that shown for the dilute solid solutions, with the added feature that there is commonly a range of intermediate concentrated solid solutions in which the thermal conductivity varies but little with composition. This is well illustrated by the typical data shown in Figs. 12 and 13. Fig. 12 shows the variation of conductivity in

copper-gold alloys, a system in which the conductivity of the two pure components is about the same and is quite high. Fig. 13 shows the variation of conductivity in the copper-nickel system, in which one component, copper, has a very much higher conductivity than the other.   The general pat-
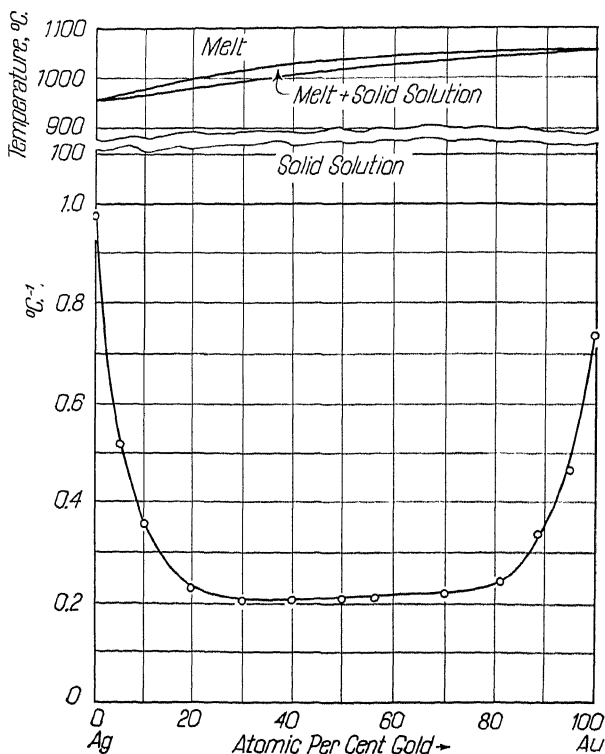
Fig. 12—Variation of Thermal Conductivity with Composition in the System Silver-Gold, Which is a Typical System in Which There is a Continuous Series of Solid Solutions. Both Components Have a High Conductivity. (After Sedstrom.)

tern of behavior is the same in each, and is quite typical of many other systems.   Addition of an alloying element lessens the conductivity of the solvent metal, markedly at first, then less effectively on subsequent additions, until the conductivity becomes almost independent of composition.   This last effect

is particularly notable in the copper-nickel system in which the conductivity is substantially constant over the range 30 to 90 per cent nickel.   The lack of sensitivity of the thermal conductivity to a change in composition is characteristic of substantially all highly alloyed metals.   A good example is the
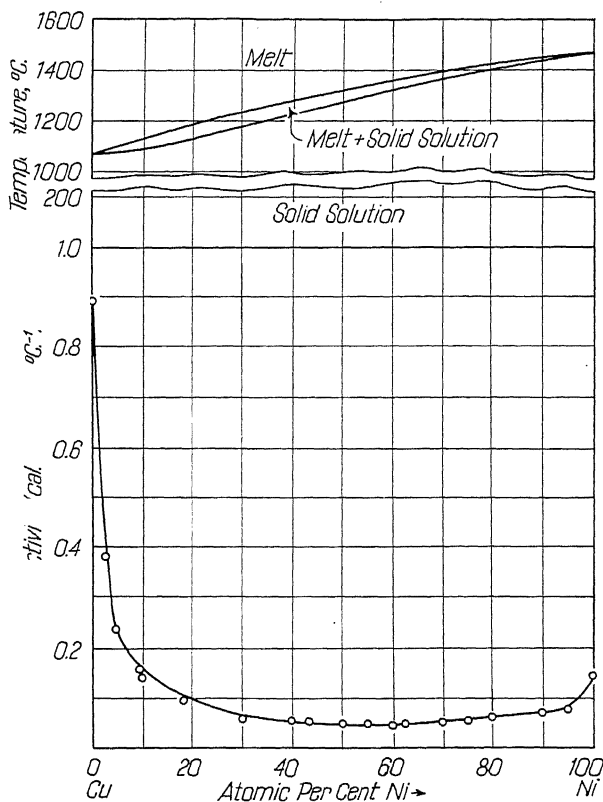
Fig. 13—Variation of Thermal Conductivity with Composition in the System Copper-Nickel, in Which There is a Continuous Series of Solid Solutions and in Which One Component is a Much Better Conductor Than the Other. (After Sedstrom.)

18-8 type iron-nickel-chromium alloy, whose conductivity does not vary much with a fairly wide variation in composition.

Effect of Ordering—The two sets of alloys just discussed are typical of systems in which there is a homogeneous ran-

dom solid solution. But in some solid solutions of the substitutional type the phenomenon of ordering may appear after certain heat treatments; when it does, it results in an appreciable change in the thermal conductivity of the alloy. In an alloy of this type, composed, let us say, of components A and B, the crystals may be much like those of A except that atoms of B have been substituted for atoms of A at cer-

$$
\begin{array}{ll}
A\ A\ B\ B\ A\ B & A\ B\ A\ B\ A\ B \\
B\ A\ B\ A\ B\ A & B\ A\ B\ A\ B\ A \\
B\ B\ A\ B\ A\ A & A\ B\ A\ B\ A\ B \\
A\ A\ A\ B\ B\ B & B\ A\ B\ A\ B\ A \\
B\ B\ A\ A\ A\ B & A\ B\ A\ B\ A\ B \\
A\ B\ B\ A\ B\ A & B\ A\ B\ A\ B\ A
\end{array}
$$

Fig. 14a   Fig. 14b

Fig. 14—Schematic Illustration of the Arrangement of Atoms A and B with (a) Disordered Solid Solution; (b) Ordered Solid Solution.

tain sites in the lattice. At high temperature this substitution is purely random, as illustrated schematically in Fig. 14a, there being no regularity in the manner in which the substituent atoms B are arranged. This random distribution is preserved by rapid cooling and, indeed, in many cases appears to be the stable arrangement at room temperature, but in some alloys of this type, if slow cooled or annealed for a long time at a relatively low temperature, the atoms rearrange themselves so that the atoms of B occupy definite sites, as is illustrated in Fig. 14b. When such a rearrangement occurs the alloy is said to be ordered, or, since the atoms of each component can be regarded as having a regular lattice of their own, the alloy is said to have a superstructure or superlattice. If an ordered alloy is reheated the ordering tends to diminish gradually until a certain definite and reproducible temperature is reached at which the remaining superstructure disappears and the alloy becomes completely disordered. This temperature can be thought of as a melting temperature for

the superlattice since it corresponds to the disappearance of the superstructure in the same way that the crystal lattice is destroyed at the ordinary melting point. The formation of a superlattice is most likely to occur when the ratio of the number of atoms of each component is a small whole number,

(a) *Random Fe Co*   (b) *Ordered Fe Co*
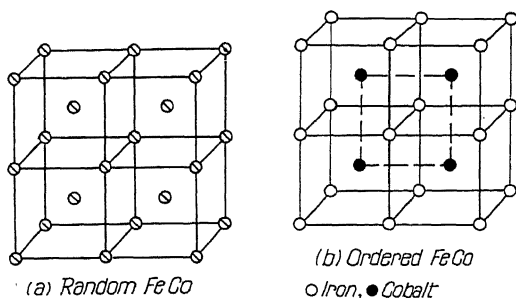
○ *Iron*, ● *Cobalt*

Fig. 15—Random and Ordered Structure in Iron-Cobalt Alloy Containing 50 Per Cent Iron. (After Ellis and Greiner.)

such as one or three, that is, ordering is most likely to occur in an alloy whose composition approaches a stoichiometric ratio such as would be expected of an intermetallic compound represented by the formula AB or A_3B. Indeed, as will be shown later, an ordered alloy possesses many of the characteristic properties of a compound.

A typical example of an ordered structure is given in Fig. 15 which shows the structure of the iron-cobalt alloy containing 50 atomic per cent iron, corresponding to the formula FeCo. Above about 730 degrees Cent. (1345 degrees Fahr.), and in any sample quenched from above this temperature, the atoms are arranged in a body-centered lattice, but with the atoms distributed randomly over the lattice points, as is shown schematically in Fig. 15a. On annealing the alloy for a long time at a temperature below 730 degrees Cent. (1345 degrees Fahr.), the structure becomes ordered, as is shown in Fig. 15b. The lattice is still body-centered cubic but the atoms are now so arranged that each iron atom

is surrounded by cobalt atoms and each cobalt atom is surrounded by iron atoms. The iron atoms form, therefore, one simple cubic lattice with the cobalt atoms arranged in a similar lattice displaced with reference to the lattice of iron atoms so that each cobalt atom comes at the center of a cube whose corners are occupied by iron atoms, whereas each iron atom lies at the center of a cube whose corners are occupied
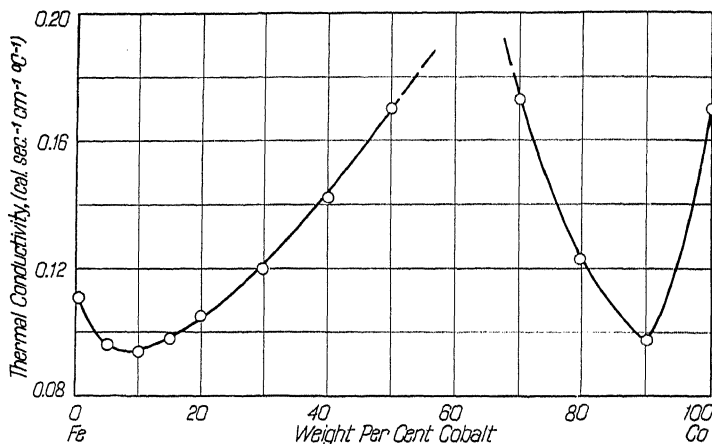
Fig. 16—Variation of Thermal Conductivity with Composition in Iron-Cobalt Alloys. (After Honda.)

by cobalt atoms. There are thus two superposed lattices which give the structure an additional periodicity not found in the disordered state; hence, the mean free path of the conduction electrons, which increases with the periodicity of the lattice, is significantly lengthened with a consequent increase in thermal conductivity over that of the disordered state. This is evident from the variation in thermal conductivity in the iron-cobalt system which is shown in Fig. 16. The maximum value was not determined but the existence of a maximum is beyond question.

Another and more complex example of the effect of ordering on thermal conductivity is found in the copper-palladium system, data for which are shown in Fig. 17, in which ordering

occurs at about 25 atomic per cent palladium and again at
50 per cent palladium.   The dotted curve gives approximately
the conductivity of quenched, or disordered, alloys; the solid
curve gives the conductivity of annealed, or ordered, alloys,
and plainly shows two maxima.   The first of these comes at
about 40 atomic per cent palladium instead of 25 per cent
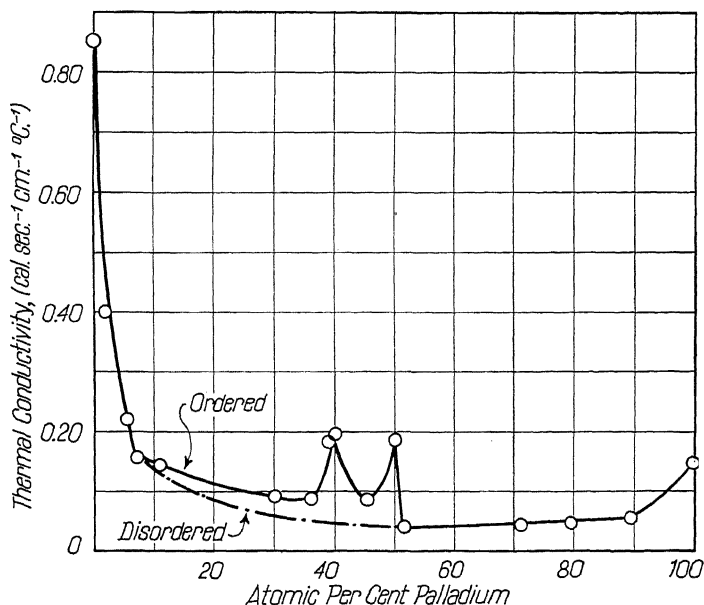
Fig. 17—Variation of Thermal Conductivity with Composition in
Copper-Palladium Alloys.   The Dashed Line Represents Disordered Al-
loys, the Solid Line, Ordered Alloys.   (Based on Data of Sedstrom.)

corresponding to the formula Cu_3Pd. This discrepancy may
be experimental error, or what is more likely, it may be due
to a failure to obtain the ordered structure in alloys contain-
ing about 25 per cent palladium, since these measurements
were made before the phenomenon of ordering was recognized.
This latter view is supported by the fact that more recent and
careful measurements of the electrical conductivity of an-
nealed copper-palladium alloys shows a definite maximum at
25 per cent palladium so that the corresponding maximum in

the thermal conductivity-composition curve should come at this composition. The second maximum occurs, as it should, at 50 atomic per cent palladium. It is interesting to note that above 50 per cent palladium no ordering has been detected.
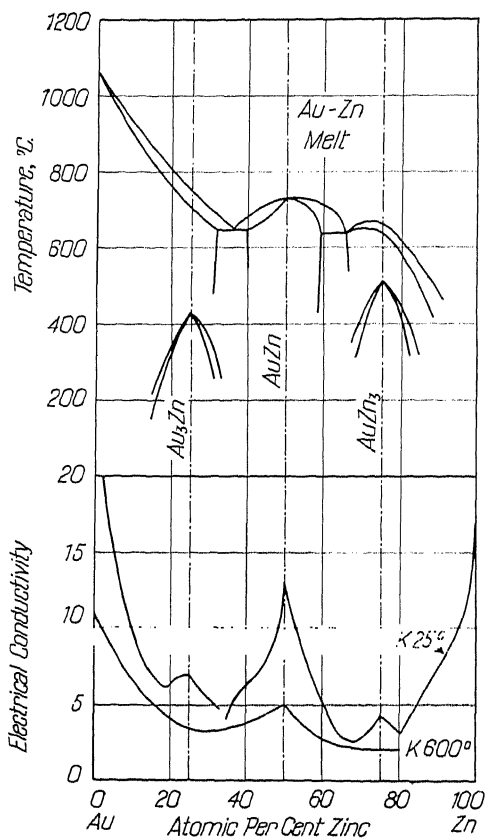
Fig. 18—Variation of Electrical Conductivity with Composition in Gold-Zinc Alloys. (After Soldau.)

Metallic Compounds—The influence of compound formation on thermal conductivity is precisely similar to that of ordering; indeed, there is some difficulty in distinguishing between many of the properties of an ordered alloy and those of a compound, and it is certain that many alloys which were

formerly considered intermetallic compounds should more properly be regarded as ordered structures. In so far as thermal conductivity is concerned a compound can be regarded as an ordered alloy whose superstructure persists up to the melting point. This is evident from the data on the thermal conductivity of gold-zinc alloys shown in Fig. 18. The upper part of Fig. 18 shows the essential features of the constitutional diagram of this system, which indicate that a type of ordering occurs in alloys containing approximately 25 atomic per cent zinc at temperatures below 425 degrees Cent. (800 degrees Fahr.), and in alloys containing 75 atomic per cent zinc at temperatures below 515 degrees Cent. (960 degrees Fahr.). On the other hand, a compound in the ordinary sense of the term, that is, an alloy whose melting temperature is a maximum in the liquidus curve and which melts at a definite temperature, appears at 50 atomic per cent zinc. The chief difference between the two types of structure is that the compound is stable up to its melting point whereas the ordered arrangement disappears at a temperature below the melting temperature of the alloy. The data on the variation of thermal conductivity with composition in this system are not very satisfactory but the electrical conductivity of carefully annealed alloys has been systematically studied over the temperature range 25 to 600 degrees Cent. (77 to 1110 degrees Fahr.) with results shown in the lower half of Fig. 18. The variation of thermal conductivity is without question similar to that shown. It will be observed that in the curve representing the data for 25 degrees Cent. there are three maxima which correspond to the two ordered structures and to the compound, the peak for the compound being the most marked. These, and other data, suggest that the effect of compound formation may in general be somewhat greater than that of ordering, but this conclusion can hardly be regarded as well-established. It should also be noted that the maxima for the

ordered structures do not appear in the curve representing
the data for 600 degrees Cent. (1110 degrees Fahr.). At this
temperature the alloys containing 25 per cent and 75 per cent
zinc are both above their critical temperature for ordering
and are therefore homogeneous random solid solutions, so that
the conductivity-composition curve is composed of two
branches, each representing the solid solution between the
compound and a pure component, similar to that for any other
such solution, as for example, those shown in Figs. 12 and 13.

As a matter of experimental observation, the thermal
conductivity of a compound is, in substantially every case,
markedly less than the conductivity of the elements of which
the compound is composed. This is to be expected on the
basis of the concept of conduction electrons since the uni-
formity of the lattice field is so much greater in the pure
metal that the electrons have a longer mean free path.

COMPLETE IMMISCIBILITY

The next type of alloy to be considered is that in which
there is complete mutual insolubility of the two components
so that the alloy consists of a mechanical mixture of two pure
metals. Such cases are by no means common but a considera-
tion of their properties makes it somewhat easier to interpret
the data for the more common type of system in which there
is limited solubility.

The electrical conductivity of an aggregate of two or
more phases has been studied fairly extensively with results
which are directly applicable to thermal conductivity. The
most fruitful of these investigations are the equations of
Maxwell and of Lichtenecker for the variation of electrical
conductivity with composition.

Maxwell calculated the conductivity of a structure com-
posed of spheres of one metal dispersed in another and arrived

at an equation which may be written

$$K_A = K_M \left(\frac{2K_D + K_M + P_D\ (K_D - K_M)}{2K_D + K_M - 2P_D\ (K_D - K_M)} \right)$$

where K_A is the conductivity of the aggregate, K_M and K_D are respectively the conductivity of the matrix material and of the dispersed phase, and P_D is the volume fraction of the dispersed
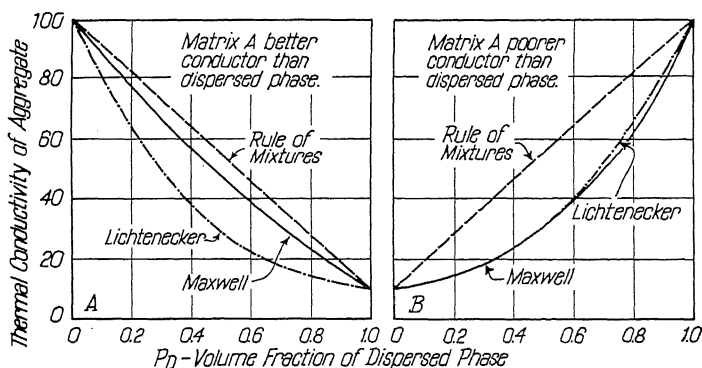
Fig. 19—Variation of Thermal Conductivity in a Mechanical Mixture of Two Insoluble Phases Whose Conductivity is 100 and 10 Units Respectively as Calculated by Means of the Equations of Maxwell and of Lichtenecker and by the Rule of Mixtures.

phase in the mixture. This relation is strictly valid only for dispersed particles of spherical shape and for relatively small values of P_D, but it gives a reasonable approximation for dispersed particles of other shapes or for the higher values of P_D. In applying this equation two types of aggregates must be considered; first, one in which the better conductor is the matrix, and second, one in which the better conductor is the dispersed material. In order to illustrate the variation in each case, two typical conductivity-composition curves are shown in Fig. 19. The curve shown in Fig. 19a is based on the assumption that the conductivity of the matrix K_M is 100 units whereas that of the dispersed phase K_D is 10; the curve in Fig. 19b is calculated on the basis that K_M is 10 and K_D is

100. These diagrams also include for comparison a dash curve, indicating what the conductivity would be if the additivity rule, or rule of mixtures, were obeyed, that is, if conductivity of a mixture composed of 50 per cent matrix material and 50 per cent dispersed phase was the sum of half the conductivity of the matrix material plus half the conductivity of the dispersed phase.

Lichtenecker has proposed an equation of the form

$$\log K_A = P_1 \log K_1 + (1 - P_1) \log K_2$$

where K_A is the conductivity of the aggregate K_1 and K_2 are respectively the conductivity of components 1 and 2 and P_1 is the volume fraction of component 1 present in the mixture. This relation is strictly valid only for a two-dimensional aggregate but the correction for a three-dimensional aggregate is so small as to be negligible in comparison with the error present in most determinations of thermal conductivity. The variation of thermal conductivity with composition for a case in which $K_1 = 10$ and $K_2 = 100$ is also shown by the dot-dash line in Figs. 19a and 19b and agrees fairly well with the curve calculated by means of Maxwell's equation for the case of $K_M = 10$, and $K_D = 100$. It should be noted that both these methods of calculation lead to the conclusion that the conductivity of an aggregate of two phases is always less than that calculated by the simple rule of mixtures, that the deviation from this simple rule is greater when the poorly conducting phase is the matrix material, and that the deviation becomes greater as the difference in conductivity between the two phases becomes increasingly greater. As the conductivity of the two components of the mixture approaches the same value, this deviation becomes less, until at the point where both components have the same conductivity, the conductivity is independent of composition and all methods agree that the conductivity-composition curve becomes a horizontal straight

line. Since in most alloys in which there is a mixture of two phases the conductivity of the two components is nearly the same, it is usually sufficient to assume that the conductivity of the aggregate is represented by a straight line connecting the points for the conductivity of the phases.

Examples of complete solid immiscibility are not common
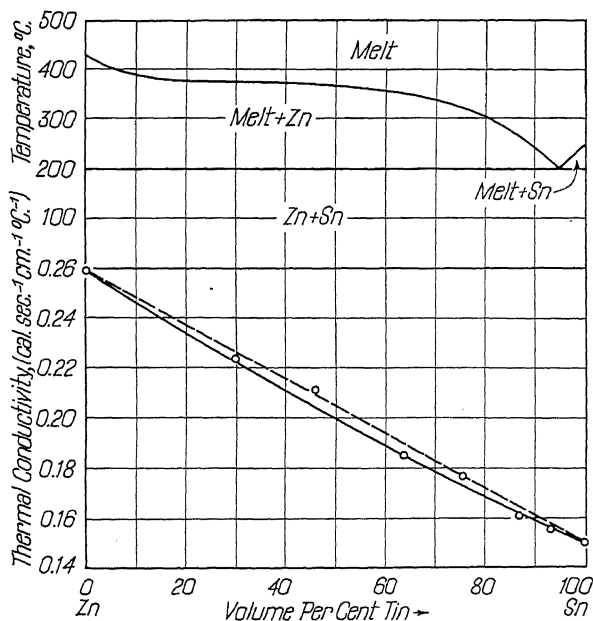
Which
F. A. Schulze.)

but there is one, the system zinc-tin, for which data are available. The data for this system, shown in Fig. 20, are not too satisfactory but suffice to show that as a first approximation the conductivity follows the additivity rule and that the deviations, which are small, are in the direction indicated by Maxwell's and Lichtenecker's equations.

Limited Solid Solubility—The variation of conductivity with composition in the common case of a system which shows

limited solid solubility is easily obtained by combination of
the variation for a solid solution with that for complete
immiscibility.   Thus, the thermal conductivity of the solid
solution decreases rapidly, with increasing alloy content, as
in the case of the iron alloys shown in Fig. 8, and when the
limit of solid solubility is reached and two phases are present,
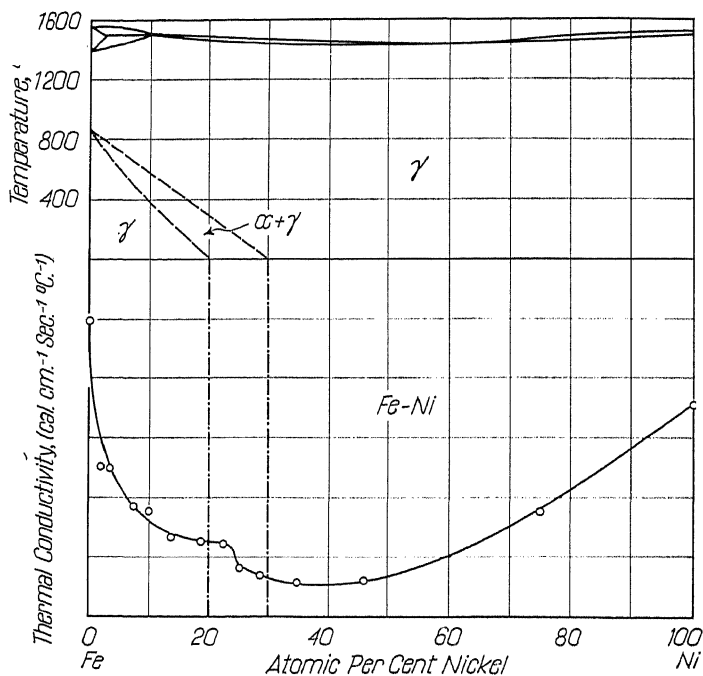
Fig. 21—Variation of Thermal Conductivity with Composition in the
System Iron-Nickel, in Which There is a Two-Phase Region Covering a
Relatively Small Range of Composition. (After Ingersoll.)

the conductivity varies approximately linearly, as shown in
Fig. 20. This type of variation is illustrated in Figs. 21 and
22.   Fig. 21 shows the variation of thermal conductivity
in the iron-nickel system in which there is a two-phase region
of very limited extent in the range 20 to 30 per cent nickel.
In this case, the two curves for the homogeneous alpha and
gamma solid solutions occupy most of the diagram and are

connected by a short line in the range of composition corresponding to the two-phase region. Fig. 22 shows the variation of thermal conductivity with composition in the tin-bismuth alloys, a system in which two phases are present over most of the composition range. Here again there are
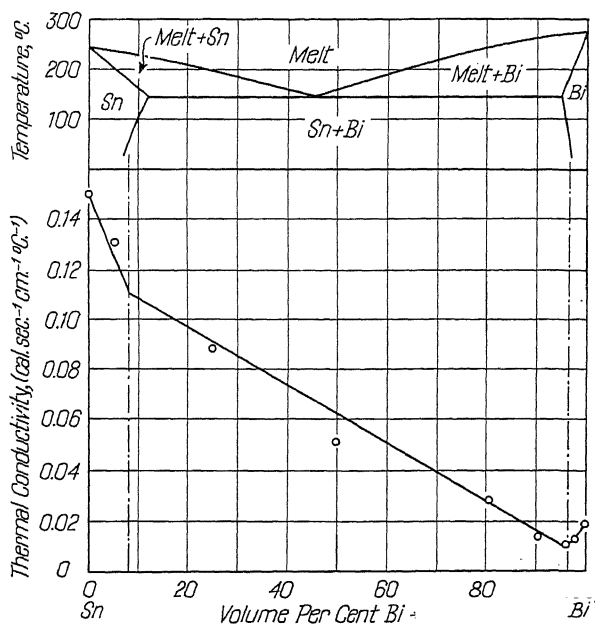
Fig. 22—Variation of Thermal Conductivity with Composition in the System Tin-Bismuth in Which the Region of Solid Immiscibility Covers a Wide Range of Composition. (After F. A. Schulze.)

the typical curves for the terminal solid solutions connected by a straight line.

Phase Changes—In addition to the effect of a change in chemical composition, the thermal conductivity of a metal or alloy may also be significantly influenced by its physical constitution, notably by phase changes, directional effects and porosity. The simplest of these is the effect of a change in phase, the most general of which is melting.

Since metallic conduction is due to the motion of electrons

Table V
Comparison of Thermal Conductivity of Solid and Liquid Metals.

Metal	Melting Temperature °C	Thermal Conductivity at Melting Point c. g. s. Units		Ratio of Conductivity of Liquid to that of Solid
		Solid	Liquid	
Mercury	−39	0.065	0.021	0.33
Aluminum	660	0.33	0.23	0.70
Zinc	419	0.21	0.14	0.65
Lead	327	0.07	0.038	0.54
Tin	232	0.14	0.079	0.56
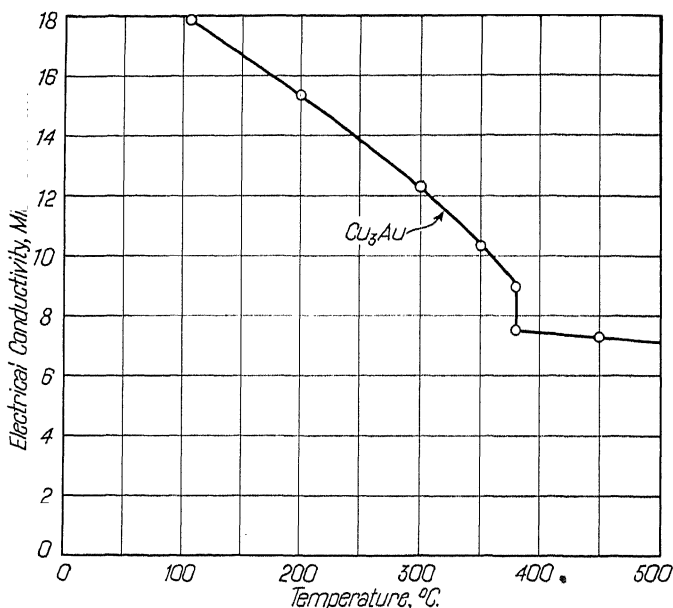
Fig. 23—Variation with Temperature of the Electrical Conductivity of an Ordered Copper-Gold Alloy of Composition Cu$_3$Au, Illustrating the Discontinuous Change in Conductivity with the Disappearance of Order. (After Sykes and Evans.)

in a periodic electric field set up by the crystal lattice, the conductivity of a liquid metal in which no lattice is present should be much less than that of crystalline metal, just as the conductivity of an alloy is greater when the structure is ordered than when it is disordered. This qualitative conclusion is amply confirmed by the experimental data given in Table V, which show that at the melting temperature the conductivity of a liquid metal is from one-third to two-thirds

that of the solid at the same temperature. There appear to be a few exceptions to this rule, notably antimony and bismuth, but they have an anomalous structure and can hardly be classed as representative metals.

By the same reasoning the conductivity of an ordered alloy should decrease at the critical temperature at which ordering disappears. Unfortunately, no adequate data on thermal conductivity are available to illustrate this behavior, but again we may fall back on the electrical conductivity. Thus in Fig. 23, which shows the temperature variation of electrical conductivity of an ordered copper-gold alloy of composition corresponding to Cu_3Au, there is a sudden decrease in conductivity at the critical temperature which is analogous to, but smaller in magnitude, than the corresponding change at the melting point.

The next type of change is the transformation of one solid phase to another, and the term phase is here used in its broadest meaning to cover not only a phase in the strict chemical sense but also the characteristic microstructures observed in alloys. In this sense, a change in phase includes not only the transformation of the body-centered lattice to the face-centered structure in pure iron but also such changes as the transformation of pearlite to ferrite and graphite. The evidence bearing on the change in thermal conductivity accompanying such a phase change is not as decisive as it is for melting but the indications are nevertheless fairly clear.

In the transformation of one crystal lattice to another, as in the transformation from alpha to gamma iron, there is no sudden change in conductivity at the equilibrium temperature as there is at the melting point or at the critical temperature for order. This is illustrated by the curve for low-metalloid iron in Fig. 24. The slope of the conductivity-temperature curve, that is, the rate of change of conductivity with temperature, changes at the equilibrium temperature but
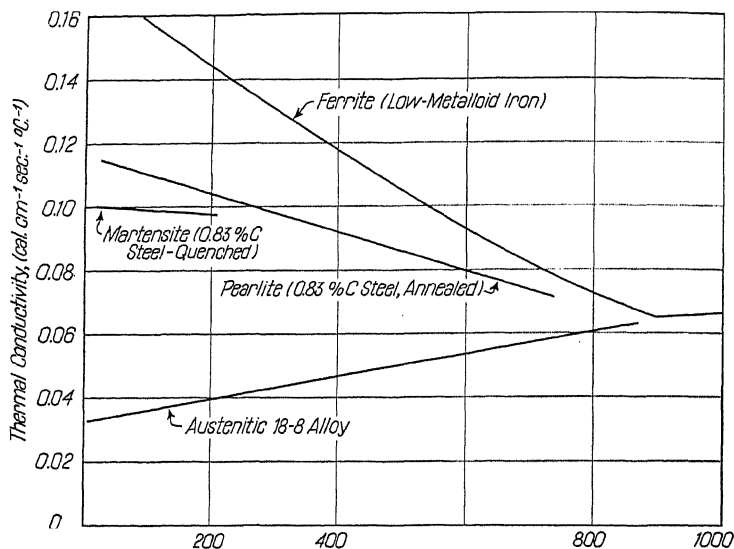
Fig. 24—The Thermal Conductivity of Ferrite, Pearlite, Martensite and Austenite.

the conductivity is the same for both phases at equilibrium. At any temperature other than that for equilibrium, the conductivity of the two phases is in general not the same, but this is largely a result of the fact that the conditions necessary to obtain one of the two phases at a temperature outside its range of stability almost always causes a change in the conductivity of the phase. For instance, Fig. 24 shows that at room temperature the conductivity of the austenitic 18-8 iron-chromium-nickel alloy is very much less than that of pure ferrite but this difference is more a result of the difference in chemical composition than of a difference in structure alone, as is illustrated by the fact that the conductivity of a ferritic 18-8 alloy is, according to Uhlig, substantially the same as that of the same alloy when austenitic.

The conductivity of an annealed steel of approximately eutectoid composition (0.83 per cent carbon) is also shown in Fig. 24, as an illustration of the conductivity to be expected

of pearlite. The conductivity is appreciably lower than that of pure ferrite because of the presence of carbon in solution in the ferrite and because of the presence of plates of poorly-conducting iron carbide. The variation in conductivity associated with an increase in the proportion of lamellar carbide is also illustrated by the data in Fig. 25, which shows the thermal conductivity of a series of carbon steels of increasing carbon content. These data are entirely in agreement with those shown in Fig. 24, in that the higher the proportion of lamellar carbide, the lower is the conductivity.

In the case of cast irons, the conductivity also changes as the pearlite is decomposed to ferrite and graphite on heating, as is illustrated in Fig. 26, which shows the percentage increase in conductivity resulting from a given decrease in the percentage of combined carbon after heat treatment.

The thermal conductivity of martensite is not well established because the specimen usually tempers during the heating necessary to make a determination, but there is no doubt that it is less than that of either ferrite or pearlite. This is clearly indicated by the data shown in Fig. 24 for a quenched sample of the same eutectoid steel used in measuring the conductivity of pearlite. The thermal conductivity of martensite is, moreover, influenced not only by its composition but also by the temperature from which the parent austenite is quenched, as is illustrated by the data shown in Fig. 27, which shows the conductivity of a steel containing 1.14 per cent carbon after being quenched from different temperatures. The conductivity decreases as the quenching temperature increases, although it should be pointed out that the conductivity scale which forms the ordinate of Fig. 27 is a very large one so that the actual change in conductivity is not so great as it might appear at first glance. This variation in conductivity probably reflects an increasing strain in the lattice with increasing temperature of quench.
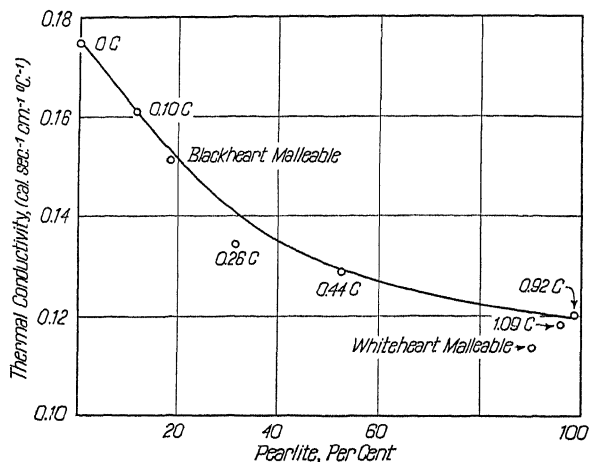
Fig. 25—The Decrease in the Thermal Conductivity of Steel with Increasing Carbon Content. (After Donaldson.)
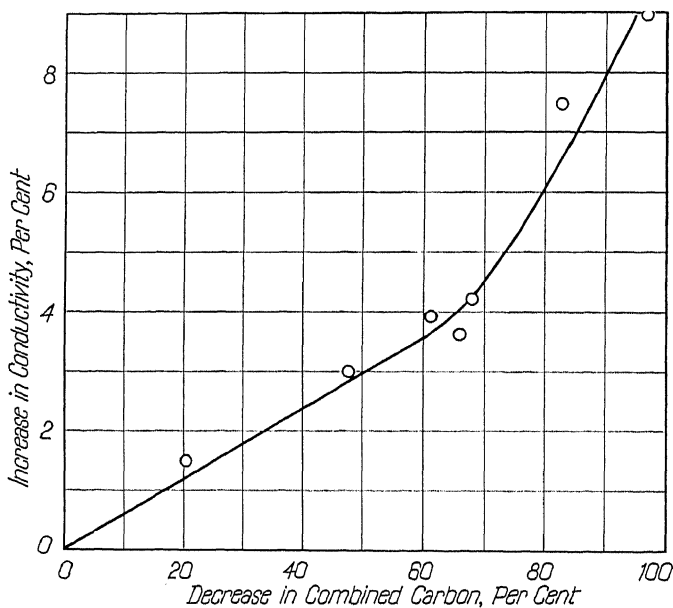
Fig. 26—The Increase in the Thermal Conductivity of Cast Iron with the Decomposition of Iron Carbide on Heat Treatment. (After Donaldson.)

To sum up briefly, the data indicate that the order of increasing conductivity is austenite, martensite, pearlite and

ferrite, the differences being greatest at room temperature and becoming much less marked at high temperatures.

CHANGES AT THE CURIE TEMPERATURE

Another type of change occurring in solid metals is the change from the ferromagnetic to the paramagnetic state at
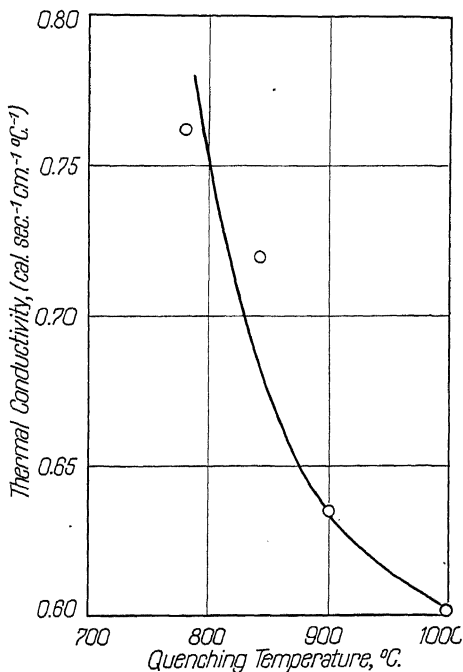
Fig. 27—The Decrease in the Thermal Conductivity of Martensite with Increasing Quenching Temperature. (After Hattori.)

the Curie point which occurs in ferromagnetic materials such as iron or nickel. The data available are again not very extensive but they suffice to show that there is no sudden change of conductivity at the Curie temperature but, in general, only a change in the slope of the conductivity-temperature curve. This is illustrated in Fig. 28 by data for iron and nickel. The change of slope in nickel is very marked, the conductivity
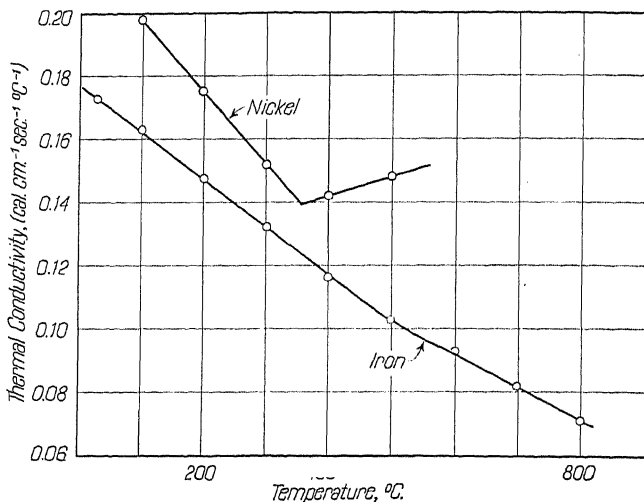
Fig. 28—The Variation of the Thermal Conductivity of Iron and Nickel at the Curie Point.

decreasing with increasing temperature below the Curie point but increasing with temperature above the Curie point; in iron, on the other hand, the change in the slope of the curve at the Curie point is hardly measurable.

DIRECTIONAL EFFECTS

Another physical factor which has an appreciable effect is anisotropic conduction in a single crystal of a metal which does not crystallize in the cubic system. This variation is of itself of limited practical significance, yet it should be considered because it has a direct bearing on the variation of thermal conductivity with grain size which is discussed in the next section. Our knowledge of conduction in single crystals is quite meager but there is no doubt that isotropic crystals, that is, those which have cubic symmetry, conduct equally well in all directions, whereas those which do not have cubic symmetry, do not conduct equally in all directions. For example, iron, which is isotropic, has the same conductiv-

ity in all directions, whereas cadmium and zinc, which crystal-
lize in the hexagonal system, have a different conductivity in
different directions, as is illustrated by Fig. 29.

For a crystal which has rotational symmetry, such as one
in the hexagonal system, the thermal conductivity in any
direction within the crystal can be expressed in terms of the
conductivity parallel and perpendicular to the axis of sym-
metry. Denoting these conductivities by $K_{||}$ and $K_\perp$ respec-

	$K_{||}$	$K_\perp$
Zn	0.241	0.255
Cd	0.286	0.229
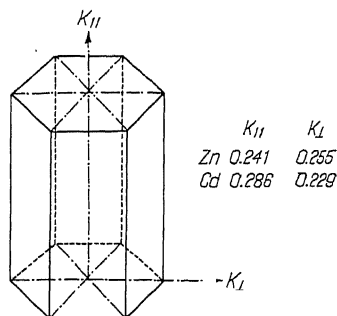
Fig. 29—The Anistropic Ther-
mal Conductivity in the Typical Hex-
agonal Metals, Zinc and Cadmium.

tively, the conductivity in any direction making an angle Θ
with the axis of rotation is

$$K_\Theta = K_\perp + (K_{||} - K_\perp) \cos^2 \Theta$$

which indicates that K_Θ varies linearly with the square of
cosine Θ. This variation is illustrated for single crystals of
tin by the data shown in Fig. 30.

The conductivity of a polycrystalline specimen of a metal
which does not crystallize in the cubic system may therefore
vary as the orientation of the grains varies, although such
variation is not likely to be large.

In addition to anisotropic conduction resulting from an
anisotropic lattice structure, there are in a practical way
other directional effects arising from the nature of the product
and these may be quite large. For example, in the annealing

of a coil of strip or a pack of sheets, heat flowing perpendicularly to the surface of the pack or the coil has to flow through a series of thermal resistances arising from the air film between the sheets, whereas, heat flowing in from the sides of the pack, or coil, has a continuous metallic path. In consequence, the effective thermal conductivity of the pack or coil
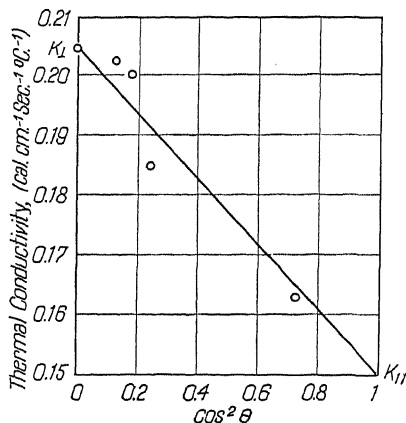
Fig. 30—The Variation of Thermal Conductivity in Single Crystals of Tin as a Function of the Angle () with the Axis of Symmetry. (Data of Bridgman.)

may be greater in a direction parallel to the surface of the sheets than in the perpendicular direction. The effect of such directionality cannot, however, in general be evaluated with any accuracy except for a few special cases.

GRAIN SIZE

For metals possessing cubic symmetry the variation of thermal conductivity with grain size appears to be negligible. Thus, direct tests by Grüneisen and Goens on gold, copper and aluminum show a maximum variation of only 4 per cent between single crystals and polycrystalline metal, an agreement almost certainly within the error of measurement. Moreover, data on steels of different grain size show no significant

Table VI
Thermal Conductivity of Steel as Influenced by Ferrite Grain Size

Steel	Open-Hearth Practice	A.S.T.M. Grain Size After Heat Treatment	Thermal Conductivity cal.sec^{-1}.cm^{-1}.°C^{-1}	Electrical Conductivity microhm^{-1}.cm^{-1}
A	Fine-grain	8	0.114 ± 0.005	0.0575 ± 0.0003
		3	0.114 ± 0.005	0.0585 ± 0.0003
B	Coarse-grain	6	0.116 ± 0.005	0.0575 ± 0.0003

variation as is evident from the data of Table VI. In this case, the evidence receives confirmation from the lack of significant variation in the electrical resistance of these same steels. For an anisotropic metal, however, the variation with grain size may be appreciable, depending on the orientation of the grains. For example, Bridgman reports that the thermal conductivity of single crystal cadmium, zinc and tin is some 20 per cent greater than that of samples of polycrystalline metal.

POROSITY

There are two types of metal products in which porosity is a factor: (a) metal powder, which has a sandy structure and in which grains of metal are substantially surrounded by air or other gas; (b) pressed and sintered metal powders which have a sponge-like structure in which pores of gas are present in a continuous matrix of solid metal. The thermal conductivity of such materials has not as yet been widely tested experimentally but it can be estimated for both types with satisfactory accuracy from Maxwell's relation for the conductivity of a two-phase aggregate (Equation 5.)

In considering an unsintered powder, if it is assumed that the metal is the disperse phase and that the conductivity of air is negligibly small compared to that of the metal, as in fact it is, then equation 5 leads to the relation

$$K_P = K_A \left(\frac{3}{P_A} - 2 \right) \tag{6}$$

where K_P is the conductivity of the powder, K_A is the con-

ductivity of air, or other gas which surrounds the particles, and P_A is the fraction of the total volume which is occupied by the air or other gas. This relation does not hold for very small values of P_A, since as P_A approaches zero, K_P becomes
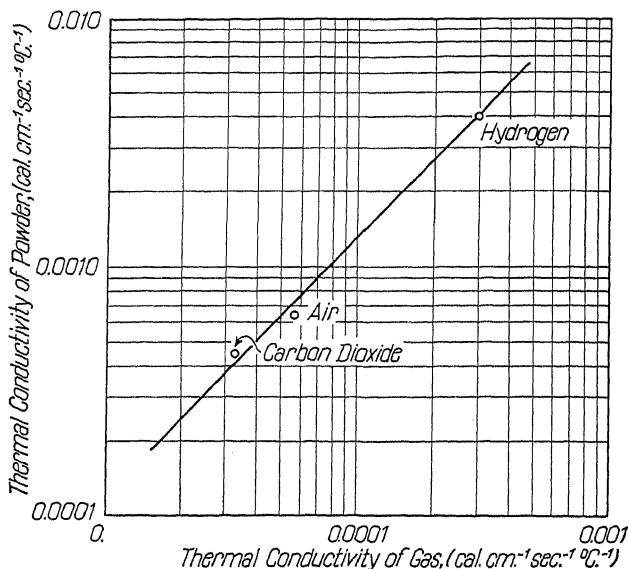
Fig. 31—Variation of the Thermal Conductivity of Fine Silicon-Carbide Powder with the Thermal Conductivity of the Surrounding Gas. (After Kannuluik and Martin.)

infinite, but it is satisfactory over the range of porosity likely to be met in practice.

It will be noted that to the degree to which equation 6 holds, the conductivity of the powder is independent of the conductivity of the metal but is directly proportional to that of the gas surrounding the particles. A good illustration of this fact is provided by the data shown in Fig. 31, which show the conductivity of fine silicon-carbide powder immersed in air, carbon dioxide or hydrogen, as a function of the conductivity of the gas. Silicon-carbide is not a metal but it shows substantially metallic conduction so that it is satisfactory for the purpose of illustration. It will be seen from

Fig. 31 that the conductivity increases directly with the conductivity of the gas. A log-log scale has been used here because of the range of conductivity involved, but the slope of the line is very nearly unity so that the variation is substantially as indicated by equation 6.

But even though the conductivity of the powder is independent of that of the metal, for practical purposes it is often convenient to refer the conductivity of the powder to that of the solid metal (K_M), that is, to write equation 6 as

$$\frac{K_P}{K_M} = \frac{K_A}{K_M}\left(\frac{3}{P_A} - 2\right) \tag{7}$$

For a ratio of $K_A/K_M = 0.001$, that is, when the conductivity of the metal is 1000 times that of air, equation 7 gives a variation of conductivity with pore volume shown by the lower curve in Fig. 32. Since this value of the ratio is about that obtaining for many alloys, and since the position of the curve is not sensitive to a change in the ratio when the ratio is as small as 0.001, the curve shown in Fig. 32 can be taken as representing the thermal conductivity of most metal powders with air as the surrounding gas. It will be noted that over most of the range of porosity found in ordinary powders, that is, for values of P_A greater than 0.50, the conductivity is not much greater than that of air. To show that this curve is of the right order of magnitude a point is included for silver powder based on the measurements of Pfleiderer. This experimental point falls slightly, but not significantly, above the curve. This curve, which applies to the ordinary metal powder, is also applicable to any metal in which the grains are surrounded by air, as for example, an 18-8 iron-chromium-nickel alloy in which the grain boundaries have been corroded away. Since such metals fall within the range in which P_A is small, that is, the range in which the curve is steep, it is evident that their thermal conductivity is extremely sensitive to intergranular attack, just as is their electrical conductivity.

In the case of a pressed and sintered metal powder the variation of thermal conductivity with porosity is very much smaller. For this type of structure, if one assumes that air is the disperse phase and that the conductivity of air is
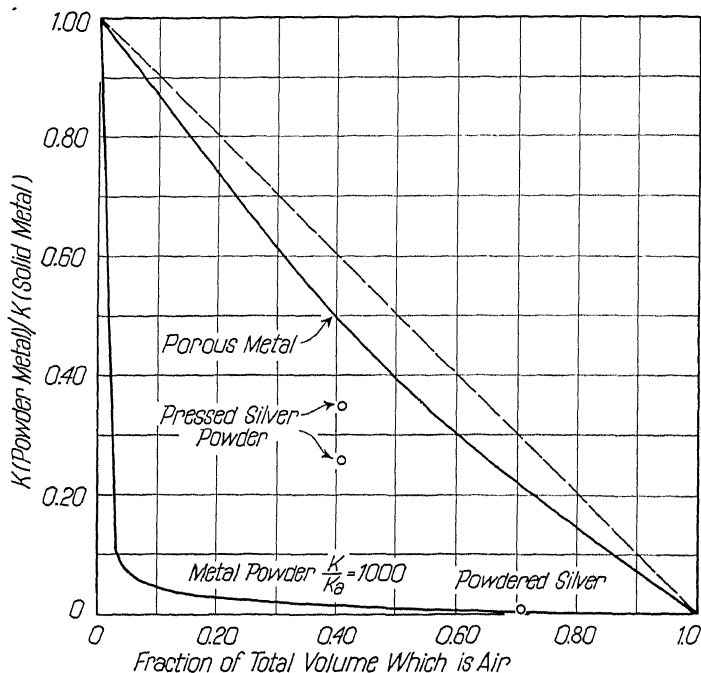
Fig. 32—The Influence of Porosity on the Thermal Conductivity. The Upper Curve Represents the Variation for a Sintered Powder, the Lower Curve Represents That of a Loose Powder.

negligible compared to that of the metal, Maxwell's relation leads to the equation

$$K_P / K_M = \frac{1 - P_A}{1 + \dfrac{P_A}{2}} \tag{8}$$

where K_P is the conductivity of the powder, K_M is that of the metal and P_A is the fraction of the total volume occupied by air. The curve corresponding to this equation is plotted as the upper solid line in Fig. 32. This curve, it will be noted, falls

somewhat, though not far, below that based on the simple additivity rule, or rule of mixtures (dotted line). Since most sintered products have a relatively low porosity, the chief interest is in that part of the curve for small values of P_A and in this range the curve is very nearly linear. Unfortunately, adequate data for checking this relation are not available but there are included in Fig. 32 two points representing Pfleiderer's measurements of the conductivity of silver powder which has been pressed, but not sintered. As expected, these points are intermediate between the curves for the loose powder and that for the sintered product.

It should also be pointed out that since the density of a powder or sintered product is directly proportional to P_A, the fraction of the total volume which is air, the variation shown in Fig. 32 applies equally to density. Thus, for a sintered product, the conductivity should decrease substantially linearly with density.

Equations 7 and 8 do not take account of such factors as pore size or the transfer of heat across pore spaces by radiation or convection. The indications are, however, that the influence of these factors is negligible, except possibly for transfer of heat by radiation at very high temperature. Even in this case, the effect seems to be small compared with the effects represented by these equations.

In practice, one occasionally encounters a virtual porosity in products other than powders. For instance, a coil of strip or a pack of sheets may be thought of as possessing lamellar pores separating the sheets. Such a structure corresponds roughly to that of a powder in which the grains are separated by air so that the variation in conductivity with the fraction of the total volume which is air will be nearly that given by equation 7, and illustrated by the lower curve in Fig. 32. This indicates that the effective conductivity of the pack of sheets in a direction perpendicular to the sheets may be only

a small fraction of the conductivity of the metal. For example, measurements on a pack of steel sheets indicate that the effective conductivity is only one-sixth that of massive steel, and layers of crumpled metal foil are used very effectively as insulating materials.

INFLUENCE OF TEMPERATURE

The discussion of the influence of temperature is confined to the range above room temperature since at very low temperatures there are a number of anomalous effects which, although of great scientific interest, have little practical significance.
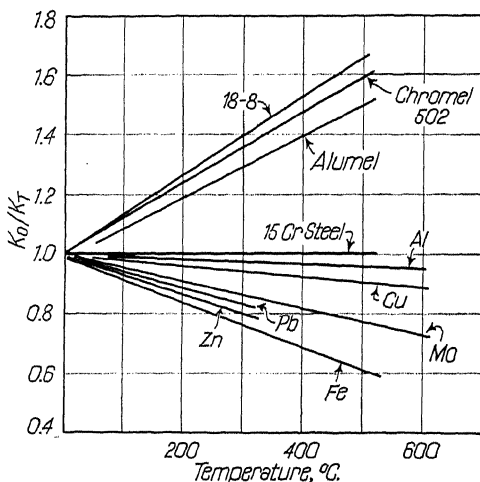
Fig. 33—The Variation with Temperature of the Thermal Conductivity of Some Typical Metals and Alloys.

The basic equation for heat conduction in metals, equation 1b (Chapter I), shows that the thermal conductivity is directly proportional to the absolute temperature and to the mean free path of the electrons, and since the mean free path tends to diminish with increasing temperature because of the increased influence of thermal agitation, the net variation in thermal conductivity is the resultant of the opposing variations

in the mean free path and in T. For pure metals, in which the mean free path is relatively long and the conductivity is high, the effect of increasing thermal agitation in most cases far overshadows the direct influence of temperature, so that, in general, the thermal conductivity of a pure metal decreases with increasing temperature. In fact, one criterion of the relative purity of two or more samples of a metal is that the sample with the greatest change of conductivity with temperature is the purest. Moreover, the variation of conductivity with temperature for a pure metal is substantially linear, as is illustrated in Fig. 33. In this Figure, the data have been plotted in terms of the ratio K_0/K_T, that is, the ratio of the conductivity at 0 degree Cent. to that at temperature T degrees Cent., so that metals of different conductivity may be conveniently plotted on the same graph. This method of plotting in effect translates all the temperature conductivity curves so that they all have a conductivity of unity at 0 degree Cent. It does not imply, however, that the conductivity of all the materials is the same at this temperature.

For alloys the situation is somewhat different. If the periodicity of the lattice is already disturbed by the presence of foreign atoms the effect of an increase in thermal agitation becomes secondary to the direct influence of temperature so that the thermal conductivity tends to increase linearly with the absolute temperature, as indicated by equation (1b) (Chapter I). This is also illustrated in Fig. 33 by the curves for the different alloys. There are, of course, intermediate cases, in which two opposing variations tend to balance each other. Such a case is the 15 per cent chromium steel shown in Fig. 33.

In going through a series of binary alloys from one pure metal to another, the slope of the conductivity-temperature curve varies regularly as is illustrated by the data for copper-nickel alloys shown in Fig. 34. Here, as in Fig. 33, the relative rather than the absolute conductivity has been used for

ease of comparison. The ordinate scale is therefore the thermal conductivity at 0 degree Cent. divided by the conductivity at temperature T, a scale which simply translates each curve so that the conductivity at zero is unity. For pure copper the conductivity decreases slightly with increasing temperature. With 20 per cent copper the conductivity increases slightly with temperature; with 40 per cent and 60
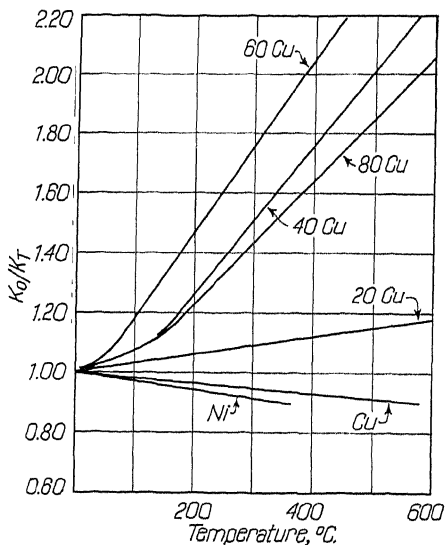
Fig. 34—The Variation with Temperature of the Thermal Conductivity of Copper-Nickel Alloys. (After Sager.)

per cent copper the increase is progressively greater but at 80 per cent copper the change of conductivity is again less, and at zero copper, that is, pure nickel, the conductivity again decreases with increasing temperature. It is to be noted that although the curves representing the conductivity of the alloys lie above those for the pure components, this does not imply that the conductivity of the alloys is greater since the conductivity scale used is again a relative rather than an absolute one so that the influence of temperature may

be emphasized.   The variation when absolute conductivity is used is similar to that shown in Fig. 35 which presents data for a group of iron alloys.

In addition to showing the same general behavior for iron alloys as is shown by Fig. 34 for copper-nickel alloys, Fig. 35 reveals the interesting and significant fact that at high temperature the conductivity of all the ferrous alloys tends to approach a common value.   Thus, although at room temperature, the conductivity of the several alloys range from 0.02 to 0.20 cal. sec.$^{-1}$ cm.$^{-1}$ °C^{-1}, at 800 degrees Cent. the spread is only from about 0.07 to 0.08 cal. cm.$^{-1}$ sec.$^{-1}$ °C^{-1}. Above 800 degrees Cent. (1470 degrees Fahr.), the conductivity of these alloys is sensibly constant, which means that at these higher temperatures the conductivity of all ferrous alloys is very nearly the same.

The behavior shown in. Fig. 35, permits one to make a fairly reliable estimate of the change in conductivity of ferrous alloys with temperature if the conductivity at some one temperature is known.   Without any serious error it can be taken that all these curves are linear and that all tend to meet at about 910 degrees Cent. (1670 degrees Fahr.).   The fact that the temperature at which these lines converge, 910 degrees Cent., is the same as that of the alpha-gamma transformation in pure iron is mere coincidence.   One can then set up an idealized chart such as is shown in Fig. 36 and by plotting the one point available and drawing a line to pass through this point and 910 degrees Cent. (1670 degrees Fahr.), a fairly good approximation to the conductivity curve is obtained.   There are a few instances where this method of estimation fails but it has proved useful and fairly reliable in a great number of applications.

In the case of compound formation, or of ordering of a solid solution, the temperature coefficient of conductivity usually changes in such a way as to become more nearly that
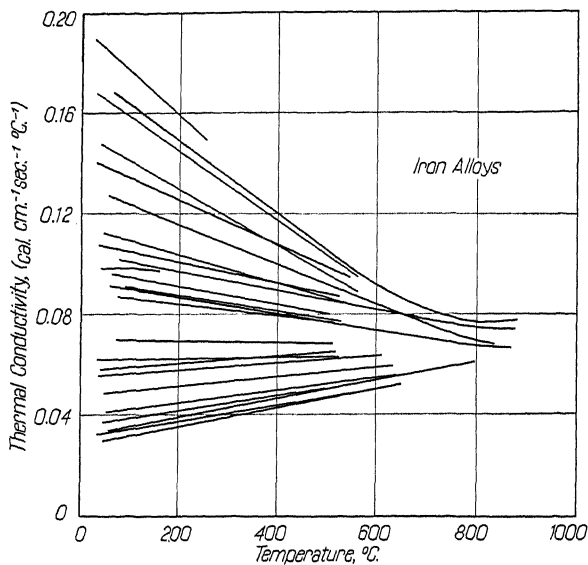
Fig. 35—The Variation with Temperature of the Thermal Conductivity of a Number of Ferrous Alloys.
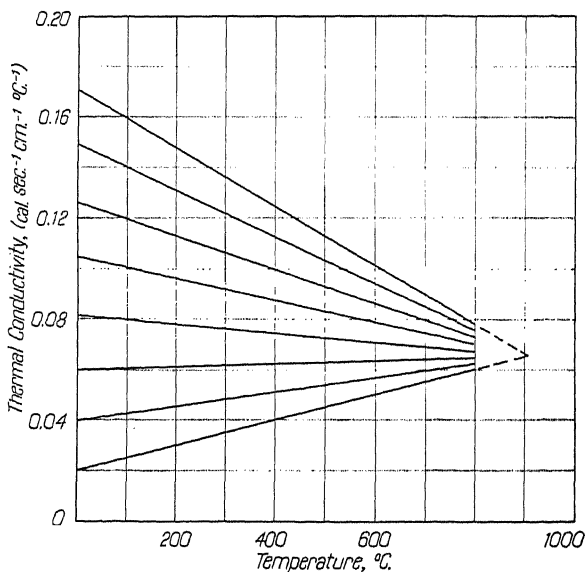
Fig. 36—Idealized form of Fig. 35 for Use in Estimating the Influence of Temperature on the Thermal Conductivity of a Ferrous Alloy.

Table VII
Influence of Tension on Thermal Conductivity

Metal	Change of Conductivity per kg/cm^2	Load in kg/cm^2
Aluminum	-3.8×10^{-6}	....
Copper	-2.1×10^{-6}	1080
Iron	-1.90×10^{-6}	2050
	-2.37×10^{-6}	1025
Palladium	-0.20×10^{-6}	770
Nickel	$+0.48 \times 10^{-6}$	1900

of a pure metal. The change of conductivity of liquid metals is in general very much less than that of solid metals.

EFFECT OF PRESSURE, STRAIN AND DEFORMATION

On the basis of the model of free electrons moving in a periodic lattice it is to be expected that any strain or deformation which distorts the lattice and the uniformity of its periodicity should lessen the thermal conductivity. This conclusion is in general confirmed by experimental observations, although the data are not conclusive.

Taking first, deformation within the elastic limit, Bridgman has measured the thermal conductivity of a number of metals in tension, with the results shown in Table VII, the thermal conductivity being diminished for every metal except nickel. It was considered in this work that the elastic limit had not been exceeded except for aluminum in which some permanent deformation occurred. In every case, however, the influence of tension is small. According to Lees and Calthrop, the conductivity of copper, aluminum and lead diminishes in torsion, the decrease being roughly proportional to the square of the angle of twist, but the measured change was so small, only a few tenths of one per cent, that it can hardly be accepted as well established. In view of these facts it may be safely concluded that deformation within the elastic limit causes no significant change in thermal conductivity.

As regards the magnitude of the effect of plastic deformation, such as cold rolling or wire drawing, it is difficult to draw

Table VIII
Influence of Hydrostatic Pressure on Thermal Conductivity

Metal	Pressure Coefficient of Thermal Conductivity $\dfrac{\Delta K}{K}/\Delta P$
Lead	$+17.3 \times 10^{-6}$
Tin	$+ 7.5 \times 10^{-6}$
Cadmium	$+ 7.4 \times 10^{-6}$
Zinc	$+ 2.1 \times 10^{-6}$
Iron	$+ 3.0 \times 10^{-6}$
Copper	$- 7.5 \times 10^{-6}$
Silver	$- 3.7 \times 10^{-6}$
Nickel	-12.0×10^{-6}
Platinum	$- 1.6 \times 10^{-6}$
Antimony	-21.0×10^{-6}
Bismuth	-31.0×10^{-6}

a definite conclusion since the experimental data are rendered questionable because of the questionable purity of the samples used, of the relatively inaccurate methods of measurement and of the influence of extraneous factors such as work hardening or phase changes. It is virtually certain, however, that for all engineering purposes, the effect of permanent deformation itself on thermal conductivity is insignificant. If such deformation is accompanied by a change in constitution of the metal or alloy or by a decrease in the degree of order, then some change in conductivity may result.

It has been definitely established by Bridgman, however, that the thermal conductivity of a metal does change when the metal is subjected to a hydrostatic pressure. His results, shown in Table VIII, indicate that there is in some cases a decrease and in others an increase with increased pressure, but again the effect is so small that it is of academic interest only.

EFFECT OF HEAT FLOW ITSELF

As another matter of academic interest, it should be pointed out that it is entirely possible that there may be a variation in thermal conductivity with the rate of heat flow, or with the temperature gradient, since the accuracy of the experimental methods does not at present permit us to decide

whether the conductivity is actually independent of these factors, as is commonly assumed. In the corresponding electrical case, the conductivity is, in agreement with Ohm's Law, to a high degree of precision, independent of the applied electromotive force, but whether the same is true of thermal conductivity and temperature gradient cannot yet be regarded as having been established beyond question.

In conclusion, of the many factors which have been shown to have an influence on thermal conductivity, chemical composition, phase changes, porosity and temperature have the greatest effect. The influence of the first three is not, in general, a matter for concern with a particular material, since we commonly deal with materials whose composition and constitution do not change radically during use, so that there is no variation in these factors and therefore no change in conductivity. Temperature, however, is a different matter since there is always a temperature drop within the metal so that in practice an average conductivity for the temperature range covered by the gradient must be used. The best average for most purposes, is simply the arithmetic mean of the conductivity at the highest and lowest temperatures involved, and since the variation of conductivity with temperature is substantially linear, the same average conductivity is obtained more simply by taking the conductivity at the average temperature.

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THE BASIC LAWS OF HEAT CONDUCTION AND HEAT FLOW IN THE STEADY STATE

The preceding chapter was devoted to a discussion of the influence of various factors upon the thermal conductivity, a quantity which was defined simply as a measure of the ability of a body to conduct heat. This rather loose definition is satisfactory for many purposes because daily experience has made us familiar with the fact that all metals do not conduct equally well under all conditions. For example, the statement that sterling silver is a better conductor than stainless steel or pewter brings to mind the experience that the handle of a silver spoon dipped into a hot liquid feels warmer than that of a stainless steel spoon of the same size dipped in the same liquid, or recalls the fact that the handle of a sterling silver tea pot is ordinarily insulated by the insertion of short sections of an insulator such as ivory, whereas this precaution is quite unnecessary in a pot made of pewter. But when we come to use the notion of thermal conductivity quantitatively, as for example, in engineering calculations, it is necessary to define it more carefully.

The precise definition is contained in the basic relation for the flow of heat in solids which states that at any given instant the rate of flow of heat past a plane section of the solid is proportional to the area of the section, taken at right angles to the direction of flow, and to the instantaneous temperature gradient at that section. This is illustrated schematically in Fig. 37. If we have a solid of any size or shape, such as the irregular body shown in Fig. 37, through which heat is flowing in the direction shown by the arrows, then the rate of flow past a plane section perpendicular to this direction is proportional to the area, A, of the section and to

the instantaneous temperature gradient at that section. The
temperature gradient at that point is the slope of the curve
representing the distribution of temperature along the path
of flow, as is illustrated in Fig. 37. Thus, for the distribution
curve shown, the temperature gradient at point X^1 is the slope
of the curve, or the tangent to the curve, at that point; or it
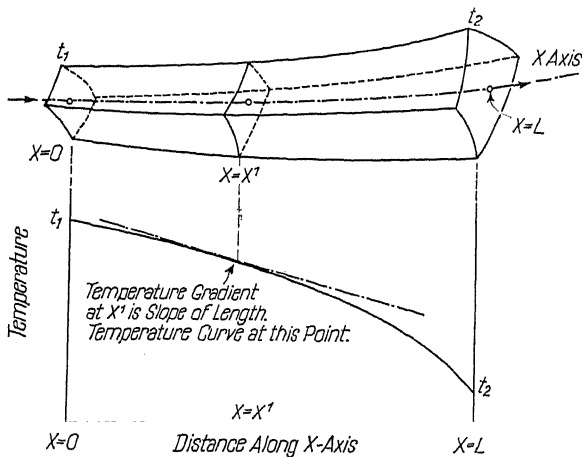
Fig. 37—Schematic Illustration of the Basic Law
of Heat Conduction.

may be considered as the temperature drop across a section
of infinitesimal thickness. The basic relation may therefore
be expressed as:

Rate of heat flow past section X^1 = K x area of section x
slope of temperature distribution curve at X^1.

In this form K is a proportionality constant which is
called the thermal conductivity and is a characteristic prop-
erty of the solid through which the heat is flowing. This basic
relation, which is the foundation of all the elaborate mathe-
matical theory of heat flow, is not usually considered as an
experimental law but should properly be so regarded since
its validity rests entirely upon experimental confirmation.

Two features of this law are worthy of special note.

First, the thermal conductivity is by definition merely a proportionality constant valid for a particular body under a particular set of conditions and although it will always have the same value under the same circumstances it may have quite a different value under another set of conditions as was amply demonstrated in Chapter II.

The second point is that this basic relation involves only the temperature gradient, or a difference in temperature, and not temperature directly; indeed, temperature itself is of no significance except for its influence on the value of the thermal conductivity. This is entirely analogous to the corresponding hydrodynamic case in which the rate of flow of water depends on the difference in level or head under which the flow occurs and does not depend upon whether the flow occurs at sea level or at a high altitude, except for second order effects such as that which might arise from a change in the viscosity of water with change in temperature between the two locations.

In applying this basic law two different types of flow must be considered: (a) that in which temperature equilibrium has been established so that the temperature of every point within the body is constant with time, and (b) that in which temperature equilibrium has not been attained so that the temperature of a given point within the body varies with time. The first type is called heat flow in the steady state; it can be treated simply and accurately but is not as common in thermal circuits containing metals as the second, and more complex type of flow, in which the temperature varies and which is called heat flow in the unsteady state. Moreover, in the steady type of flow it so happens that in most cases the thermal conductivity of the metallic portions of the circuit has relatively little influence on the total rate of heat transfer, a fact which is not nearly so widely recognized as it should be, and which deserves considerable emphasis. The remainder of this chapter is therefore devoted to a discussion of heat flow

in the steady state and to a consideration of the circumstances under which the thermal conductivity of the metallic parts of the circuit is significant, leaving the discussion of the unsteady state for the final chapter.

Heat Flow in the Steady State

Since for heat flow in the steady state in a homogeneous solid of constant cross section, the temperature at any point, and in consequence, the rate of flow of heat past that point,

Rad of Cross Section A and Thermal Conductivity K

Temperature

Distance Along Rod
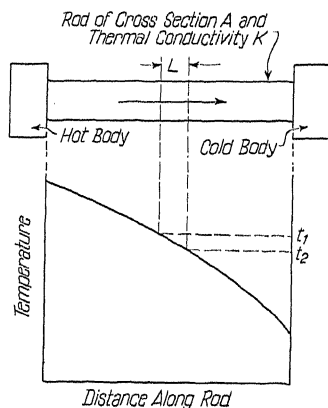
Rate of Heat Flow Through Section L is:

$$q = \frac{KA(t_1 - t_2)}{L}$$

$$= (t_1 - t_2) \times \text{Conductance of Section}$$

$$= \frac{(t_1 - t_2)}{\text{Resistance of Section}}$$

Fig. 38—Schematic Illustration of Heat Flow in the Steady State.

is constant, the restriction of considering a plane section may be removed and we may consider a portion of the body, or even the body as a whole. If the faces of the body to which, and from which, the heat flows are substantially at uniform temperature and if the heat flow is entirely normal to these surfaces, the rate of flow through the body is equal, as we have seen, to the product of the thermal conductivity, the

constant area of the heat flow path, and the temperature gradient through the body, which, since we are no longer considering a plane section, is the temperature difference between the hot face and cold face of the section considered, divided by the thickness of the section. The rate of heat flow is therefore:

$$q = \frac{KA\ (t_1 - t_2)}{L} \qquad (9)$$

where q is the quantity of heat flowing per unit time, K is the thermal conductivity, A the cross sectional area of the body, L is the thickness, and t_1 and t_2 are respectively the temperature of the hot and cold faces.

The significance of this equation is illustrated schematically in Fig. 38. Suppose that we have a rod of copper 10 centimeters long and 0.5 square centimeters in cross section which is heated at one end to 100 degrees Cent. and cooled at the other to 0 degree Cent. The thermal conductivity of copper is 0.934 cal. cm.$^{-1}$ sec.$^{-1}$ °C.$^{-1}$ The rate of flow of heat along the bar is therefore

$$q = \frac{0.934 \times 0.5 \times (100° - 0°)}{10} = 4.67 \text{ calories per second}$$

If the temperature range had been much greater so that the variation in thermal conductivity with temperature should be taken into account, then the mean conductivity for the temperature range t_1 to t_2 should have been used.

The thermal conductivity, K, is, as we have seen, a property characteristic of a given material under a given set of conditions and is a measure of the ability of the material to conduct heat under certain specified conditions; it is a property which is entirely independent of the size and shape of the body through which the heat is flowing. The measure of the ability of a body of given size, such as the length of rod, L, in Fig. 38 to conduct heat is given by the ratio KA/L,

which is called the thermal conductance of the body and is designated by C. Thermal conductivity is therefore a general property characteristic of a given material; thermal conductance is a quantity characteristic of a particular piece of the material. Thus, the thermal conductivity of copper is 0.934 cal. cm.$^{-1}$ sec.$^{-1}$ °C.$^{-1}$; the conductance of the rod of copper just considered, that is, a rod 10 centimeters long and 0.5 square centimeters in cross section, is

$$\frac{0.934 \times 0.5}{10} \text{ or } 0.0467 \text{ cals. sec.}^{-1} \text{ °C.}^{-1}$$

The reciprocal of the thermal conductivity $1/K$ is called the thermal resistivity. It is, like the conductivity, a specific property of a material, a measure of the ability of that material to resist the flow of heat and is entirely analogous to electrical resistivity. The thermal resistance of a particular body of given dimensions, such as the rod in Fig. 38, is L/AK and is designated by R; it is entirely analogous to the electrical resistance of the body. For example, the thermal resistivity of copper is $1/0.934 = 1.07$ cal.$^{-1}$ cm. sec. °C.; the thermal resistance, R, of the rod of copper considered above is $L/AK = 10/0.5 \times 1.07 = 21.4$ cal.$^{-1}$ sec. °C. Resistivity and resistance are much more commonly used than conductivity and conductance in electrical measurements and it is strange that they have not been more widely used in heat flow since they have distinct advantages in many cases.

Expressing the flow of heat in terms of resistance, equation (9) becomes:

$$q = \frac{(t_1 - t_2)}{R} = \frac{\Delta t}{R} \tag{10}$$

This, it will be observed, is identical with Ohm's law for the flow of electricity, namely, that the rate of flow of current (amperes) is equal to the potential or voltage drop divided by the resistance. Indeed, it was the application of equation

(10), originally formulated by Fourier for flow of heat, to electrical circuits which has made the name of Ohm famous. It is interesting to note that whereas in the flow of electricity, there is a unit, the ampere, for rate of flow, no corresponding unit for thermal flow has yet been widely used.

It is also to be noted that all these quantities, thermal conductivity, resistivity, conductance and resistance, characterize only the transfer of heat within the boundaries of the material or body in question and are independent of the means by which heat is imparted to or extracted from the surfaces of the material. In practice, however, the simple case of heat transmission through a single homogeneous metal is rarely, if ever, met and it is usually necessary to take into account the means by which heat is imparted to or extracted from the metal; that is, the whole thermal circuit must always be considered just as a complete electrical circuit must be used in studying the flow of electricity. For this reason it is desirable to review the laws of thermal circuits.

LAWS OF THERMAL CIRCUITS

When heat flows through two or more bodies connected in parallel, as shown in Fig. 39, each may be considered independently of the others so that the total rate of flow is the sum of the rates of flow through the several thermal resistances. The rate of flow, q, through any one body is

$$q = (t_1 - t_2)\frac{KA}{L} = (t_1 - t_2)\,C = \frac{(t_1 - t_2)}{R} \qquad (11)$$

that is, it is equal to the total temperature drop multiplied by the conductance or divided by the resistance of the body. The aggregate heat flow, q_A, through several bodies, designated as a, b, c, etc., respectively, is then

$$q_A = q_a + q_b + q_c + \ldots\ldots \qquad (12)$$
$$= \Delta t_a\,C_a + \Delta t_b\,C_b + \Delta t_c\,C_c + \ldots\ldots \qquad (13)$$
$$= \frac{\Delta t_a}{R_a} + \frac{\Delta t_b}{R_b} + \frac{\Delta t_c}{R_c} + \ldots\ldots \qquad (14)$$

where Δt is the temperature drop through each body. If on the other hand, the temperature drop is the same for each of the bodies a, b, c, etc., that is, $\Delta t_a = \Delta t_b = \Delta t_c$ then

$$q_A = \Delta t \; (C_a + C_b + C_c + \ldots\ldots) \qquad (15)$$
$$ = \Delta t \; (1/R_a + 1/R_b + 1/R_c + \ldots\ldots) \qquad (16)$$

Thus, for parallel flow under the same temperature gradient the conductance of the several branches is additive; that is, the conductance of the circuit is the sum of the conductance

$$\Delta t = t_1 - t_2$$
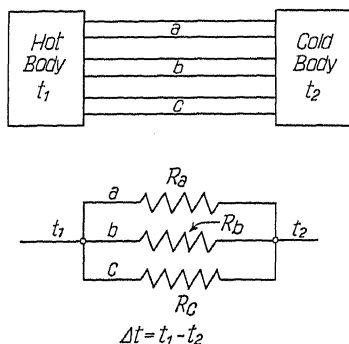
Total Rate of Heat Flow is:

$$q_A = \Delta t (C_a + C_b + C_C)$$
$$= \Delta t \left(\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_C} \right)$$

Fig. 39—Schematic Illustration of Heat Flow in Parallel.

of the branches. It should be noted that these equations are identical with those for parallel flow of electricity and, indeed, a thermal circuit can be represented schematically in the symbols commonly used for electrical circuits, as is illustrated in Fig. 39. Heat flow in parallel through circuits of which metals are a part is rather rare, although it does occur in such cases as flow through cooling fins.

More common is heat flow in series, in which the same thermal current flows through the whole circuit, as is shown in Fig. 40. In this case the rate of flow can be calculated:

(a) by multiplying the temperature drop through any homogeneous part of the circuit by the conductance of this section; (b) by dividing this temperature drop by the resistance of this part of the circuit; or (c) by dividing the total temperature drop by the sum of the individual thermal resistances. Thus:

$$q_A = \frac{(t_1 - t_2)_a}{R_a} = \frac{(t_1 - t_2)_b}{R_b} = (t_1 - t_2)_a \, C_a = (t_1 - t_2)_b \, C_b \quad (17)$$

$$\frac{(\Delta t)_A}{R_A} = \frac{\Delta t_a + \Delta t_b + \Delta t_c +}{R_a + R_b + R_c + \ldots\ldots} \quad (18)$$

where the subscripts a, b, and c refer to the resistances comprising the circuit and the subscript A refers to the aggregate temperature drop or resistance. In series flow it is resistance
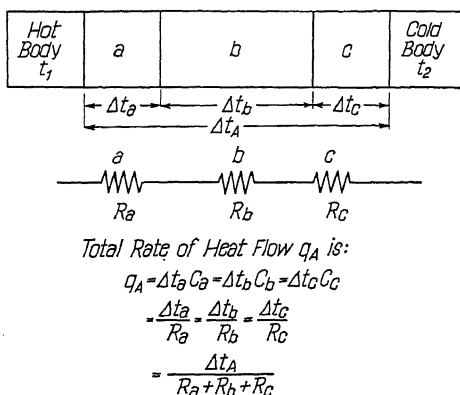
Fig. 40—Schematic Illustration of Heat Flow in Series.

which is additive so that the total resistance is the sum of the individual resistances as is illustrated schematically in Fig. 40. The above equations, which are identical with those for series flow of electricity, can be written in terms of conductance but are much more cumbersome in this form.

A typical case of series flow is the passage of heat through a composite metal such as mild steel clad with a layer of stainless alloy. Taking the thermal conductance of mild carbon steel as 0.14 cal. cm.$^{-1}$ sec.$^{-1}$ °C.$^{-1}$ and that of stain-

less steel alloy as 0.05, it is instructive to compare the effective conductivity of a unit thickness of metal made (a) entirely of mild carbon steel; (b) with 10 per cent of the thickness stainless alloy; (c) with stainless alloy making up 20 per cent of the thickness. These structures are illustrated in

All Mild Carbon Steel
K=0.14
a.)

10% 18-8
K=0.13
b.)

20% 18-8
K=0.12
c.)
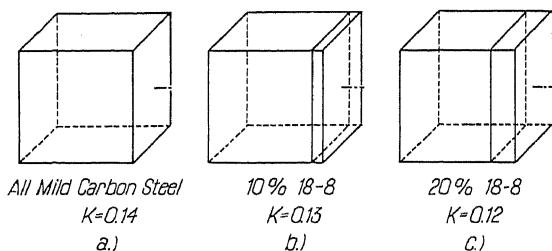
Fig. 41—The Thermal Conductivity of a Composite Metal Composed of Mild Carbon Steel with an Over-lay of 18-8 Stainless Alloy.

Fig. 41. Now the conductivity is in effect the conductance of a unit cube under unit temperature gradient; hence, we can make the comparison most readily for a cube of this kind. The simplest method is also to use resistance since it is additive in series flow. The resistance of a unit cube of mild carbon steel is $1/0.14 = 7.18$; that of the stainless alloy is $1/0.05 = 20$. Now in cube (b), since the thickness of metal is nine-tenths mild carbon steel and one-tenth stainless, the total resistance is $0.9 \times 7.1 + 0.1 \times 20 = 6.48 + 2 = 8.39$, and the conductivity is $1/8.39$ or about 0.12. Similarly, the material with 20 per cent stainless has a conductivity of $0.8 \times 7.1 + 0.2 \times 20 = 9.76$, or a conductivity of $1/9.76$ or 0.105. Direct measurement on such materials by Rädeker and Schöne gave 0.13 and 0.12 respectively.

RELATIVE INFLUENCE OF METALLIC RESISTANCES IN THERMAL CIRCUITS

As already pointed out, the thermal conductivity, or the thermal resistivity, is a specific property of a given material,

characterizing only the transfer of heat within the boundaries of that material, and is independent of the means by which the heat is imparted to or extracted from the body. Rarely, if ever, however, does one have to deal with heat flow through a single homogeneous metal. In substantially every case heat flows from some medium to the metal or from the metal to some other medium and in such cases there is a series of thermal resistances, and under the law of series flow the net rate of heat transfer is controlled by the resistance of the whole thermal path. For example, certain metals, notably silver and copper, have a very high thermal conductivity, that is, they possess a large potential capacity for conducting heat, but they must be supplied with all the heat they conduct, and this limitation sets a practical limit to the extent to which the high conductivity of these metals can be utilized.

A glance at the equations governing series flow shows that any portion of the thermal path which has a relatively high conductivity, that is, a relatively low resistance, has but little effect on the rate of flow, which is substantially controlled by the resistance of those parts of the circuit which are the poorest conductors. Indeed, a section of high conductance, or low resistance, can usually be removed or omitted from the thermal circuit without appreciably altering the rate of flow, but in no case can the insertion of a material of low thermal resistance accelerate the flow; heat flows but cannot be pushed.

In considering thermal circuits of which metals form a part, the main interest is, paradoxically, not in the metallic, but in the nonmetallic portions of the path; for the resistance of the nonmetallic sections is commonly many times greater than that of the metals. In addition to the obviously "nonmetallic" portions of a circuit, for example, insulation, or visible scale or other film, there are other, frequently less evident but equally significant, resistances in the form of

contact resistances between solid bodies and of film resistances which exist at the interface between a solid and a gas or liquid. The interface between two metals always has an appreciable resistance even though care is taken to assure good contact. This is well illustrated in Fig. 42 which has been prepared from the data of Jacobs and Starr on the
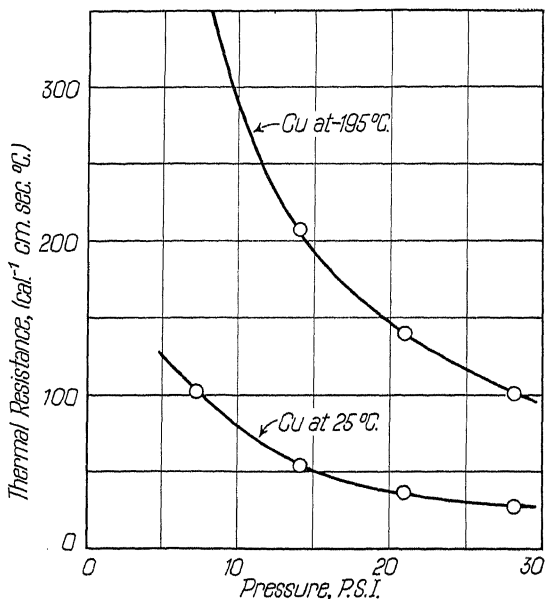
Fig. 42—The Influence of Pressure Upon Thermal Contact Resistance Between Two Polished Surfaces of Copper. (After Jacobs and Starr.)

variation with applied pressure of the thermal conductance of a contact between two carefully polished optically-flat copper surfaces in vacuum. The lower curve (for 25 degrees Cent.) shows that at a pressure of 30 pounds per square inch the resistance was about 25 cal.$^{-1}$ sec. °C which is a little greater than that of a copper rod 10 centimeters long and 0.5 square centimeters in cross section; at the lower pressures this resistance rises rapidly to over 10 times this amount. The data also give an idea of the effect of adsorbed gas films

on the surfaces of contact in that the higher resistance at —195 degrees Cent. (see upper curve in Fig. 42) is due chiefly to the presence of a very thin gas film adsorbed on the contact surfaces at this low temperature even in a fairly good vacuum. It is evident that even under conditions favoring good contact the contact resistance is often large. When it is considered that in ordinary metallic contacts, the surface film is likely to be thicker, the contact surfaces are not polished and flat but are rough, it is clear that contact resistances may become very large indeed.

One exception, however, is the contact between backing and overlay in a clad metal such as the stainless-clad mild carbon steel just discussed. In such composites, the adherence at the boundary is so complete and the contact so intimate that there is no significant thermal resistance at the interface. This is illustrated by the data just given which show that the experimentally determined values agree quite well with those calculated on the assumption that the contact resistance is negligible.

In addition to contact resistance between two solids, there is another resistance, which is equivalent to a contact resistance, between a solid and a liquid or gas. For whenever a liquid or gas is in contact with a solid there appears to be an adherent and relatively stationary film of fluid on the surface of the solid, a film which becomes thinner as the velocity of the fluid parallel to the surface increases but which breaks away from the solid only at exceedingly high velocities, if at all. Through such a film heat is transmitted by conduction only. Since most gases and liquids are relatively poor conductors of heat, there results a large resistance to heat flow at the surface of contact, accompanied by a large drop in temperature at the interface. A typical example of such a series of resistances is shown schematically in Fig. 43. In considering such films, the assumption is made that the

thickness of the film is definite and reproducible, though unknown, and the conductance of the film is expressed in terms of a "film coefficient", usually designated by h, which combines the factors of conductivity and thickness. Thus, the coefficient is expressed as cal. sec.$^{-1}$ cm.$^{-2}$ °C^{-1}, or Btu. ft.$^{-2}$ sec.$^{-1}$ °F^{-1}, which omits the element of thickness.
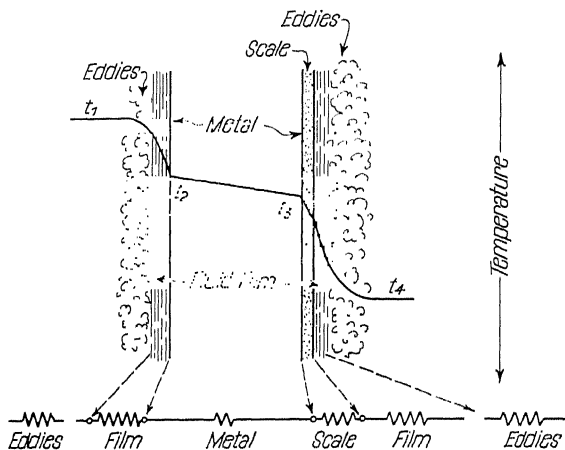
Fig. 43—Schematic Illustration of the Several Thermal Resistances Encountered in Transferring Heat from One Fluid Through a Scaled-Metal Wall to Another Fluid.

The exact evaluation of a film coefficient is far from simple since the resistance varies with many factors, notably the physical properties of the fluid, the nature and shape of the surface, and the velocity of the fluid as it flows past the surface of contact. In many cases, indeed, it can be estimated only roughly. A detailed discussion of the evaluation of such film coefficients is quite beyond the scope of these lectures, but since the resistance to heat flow of such films, is frequently the largest and the controlling factor in many cases of heat transfer, they deserve some attention.*

One of the highest of such resistances is that at the

*The interested reader will find a fairly complete discussion of film coefficients in such standard texts as "Heat Transmission" by W. H. McAdams, McGraw-Hill Co., N. Y. (1933).

boundary of a solid and a gas. For example, the film coefficient, h_G, for heat transfer from a vertical wall to "still air"; that is, air stirred only by natural convection is:

$$h_G = 0.7 + \frac{\Delta t}{375} \tag{19}$$

where h_G is in Btu. hr.$^{-1}$ ft.$^{-2}$ °F^{-1}, and Δt is the difference in temperature in degrees Fahr. between the metal and the ambient air.   This relation holds for values of Δt between 10 and 500 degrees Fahr.   For example, if $\Delta t =$ 375 degrees, $h_G = 1.7$.   Comparing this value with the conductance of a plate of iron $\frac{1}{4}$ inch thick whose conductance per square foot of area is 480 Btu. hr.$^{-1}$ ft.$^{-2}$ °F^{-1}, it is evident that the conductance of the film is some 300 times less than the conductance of the metal; that is, the film has a resistance 300 times that of the metal.   Clearly, in such a case, the conductance of the metal is of little significance.

For convection upwards from horizontal walls, h_G increases by 10 per cent; for convection downward from horizontal walls it is decreased by 50 per cent.   For air in motion, h_G is given by a complicated expression involving the specific heat of air, and the mass and velocity of the fluid.

For heat transfer from solids to liquids stirred by natural circulation, h varies from 10 to 300 Btu. hr.$^{-1}$ ft.$^{-2}$ °F^{-1}, from solids to boiling liquids it varies from 200 to 4000 depending on the conditions; for condensing steam on a solid, h $= 2000$. In general the most effective means of increasing h, that is, of lowering the film resistance, are those which reduce the thickness of the film.   Thus, a rapid movement of the fluid past the solid surface can greatly decrease the film resistance by tearing off the film.

In the important special case of heat transmission into a boiling liquid, the formation and disengagement of bubbles on the heating surface, as well as the convection currents in the liquid mass, both aid so greatly in the diminution and

rupture of the surface film that the surface coefficient for heat transfer into a boiling liquid is sometimes six-fold that for transfer into the same liquid below its boiling point. On the other hand, if the temperature difference between the surface and the boiling mass be slight, it is possible to form a more or less continuous film of vapor on the heating surface, the disengagement of which is very slow, and which acts as an insulating layer between the solid and liquid. Under these conditions the coefficient can be very low, but will rise rapidly with increasing temperature difference and the consequent stirring produced by the rapid evolution of the bubbles.

It should also be kept in mind that the rate at which heat is imparted to or extracted from the surface by radiation has a significant effect on the total heat flow. Thus, an oxidized iron surface will absorb or radiate heat at a very much greater rate than a surface of bright silver or copper. Heat transfer by conduction through a metal, may therefore be significantly influenced by the effects of radiation or convection at the surface.

The existence of these high contact and film resistances and the influence of heat transfer to and from metallic surfaces by radiation and convection makes it virtually impossible to employ the relatively great conductivity of metals to any but a limited extent in heat flow in the steady state, even when a high rate of heat transfer is desired. For this reason the thermal conductivity of a metal is not often a property which has a decisive influence on the choice of a metal in the design and construction of apparatus or machinery in which heat flow is involved. It cannot be emphasized too strongly that it is useless to reason offhand that a metal of high conductivity can be used to give a high heat transfer, or that a metal of low conductivity can be used to decrease heat transfer. The chances are that any increase or decrease so obtained will be insignificant except for a very few cases.

It will be of interest to give a few actual illustrations. The simplest and most extreme case is that of heat flow from gas on one side of a metal sheet to gas on the other side. Under any condition met with in practice, the thermal resistance of the sheet, regardless of the kind of metal of which it is composed, is utterly insignificant compared with that between the metal surface and the ambient gas on the two sides. While the gas on one side may be many degrees hotter than that on the other, the temperature drop through the metal itself will be only a small fraction of a degree, as is illustrated schematically in Fig. 43. An attempt to increase the heat transfer appreciably by using a metal of high conductivity is obviously futile. As a matter of fact, it so happens that the heat transfer might be materially reduced by using a metal sheet composed of silver, the best conducting metal known. The increase in heat transfer due to the better conducting properties of silver would be insignificant, but the decrease due to the fact that a bright silver surface has a very low emissivity for radiant heat might be considerable under certain conditions.

If the metal plate is in contact with liquid on each side, the resistance to heat flow between the body of the liquid and the metal surface is in general much less than in the case of a gas, and the relative importance of the resistance of the metal itself is therefore much greater, but the latter is still practically negligible under any ordinary condition met with in present industrial practice.

Heat transfer from condensing vapor on one side of a metal wall to boiling liquid on the other, as occurs in certain types of evaporators or stills heated by steam, is very much more rapid than in the examples previously considered. Thermal resistances between the fluids and the metal surfaces are therefore small, and the resistance of the metal itself, that is, the temperature drop through the metal, is relatively

large enough to be at least considered. In this connection numerical thermal conductivity data are of interest, though by no means the determining factor in the selection of the metals to be used.

A case in point is the stainless-clad mild carbon steel which has already been discussed. The statement has been made that heat transmission is better with the clad metal than with a solid plate of the stainless alloy of which the overlay is made. Such a statement is misleading, for under many conditions there is virtually no advantage in heat flow to be gained by using the clad material. Thus, taking the thermal conductivity of the stainless alloy as 0.05 units and that of the clad steel as 0.12, there appears to be a marked difference but when one considers the whole thermal circuit including the virtual resistances to heat flow at the surfaces there is practically no difference. For example, taking a unit cube, the resistance of the alloy is $1/0.05 = 20$, that of the clad metal $1/0.12 = 8.3$. If these materials are now to be used in a process in which the surface resistances will be ten or twenty times as great as that of the metal, the rate of heat flow will be substantially identical and the clad metal has no significant advantage over the alloy. Moreover, if the metals are to be used under conditions in which an appreciable thickness of scale forms on the carbon steel but not on the stainless alloy, it is conceivable that the added resistance arising from the scale on the carbon steel side might make a significant difference and that the stainless alloy, even though it has a lower thermal conductivity, might give a greater heat transfer. This example shows that broad statements of this kind have a distinctly limited significance and that each case should be examined by itself.

Where heat has to flow along a thin piece of metal rather than through it, conductivity becomes somewhat more important. In such cases it is usually desired that the heat flow

be small rather than large. Familiar instances are found in the handles of various utensils which are heated in use. The handles may become too hot to hold if a metal of fairly high conductivity, for example, aluminum, is used, as witness the widespread use of wooden handles on aluminum frying pans.

Heat transfer in internal combustion engines appears to offer a promising field for the application of data on the thermal conductivity of metals. In such engines large amounts of heat must necessarily be conducted through metals and dissipated externally.

In air-cooled engines, the heat is dissipated to the external air by means of metallic fins which are usually integral parts of the block and cylinder head. The rate of dissipation is dependent to an appreciable extent upon the thermal conductivity of the metal of which the fins are composed, but this property is not the determining factor in the choice of metals. The thermal conductivity is, however, of considerable interest in analyzing the performance of air-cooled engines, and may be of importance in the consideration of possible fin design.

Heat transferred directly from the hot gases in the cylinder to the piston head flows almost entirely through the oil film into the cylinder wall. The metallic path is of some length, extending from the center of the piston head outwards and down the skirt of the piston. The oil films, particularly at the rings, are very thin and offer little resistance to heat flow. The metallic portions of the path for heat flow are therefore of considerable importance, the conductivity of the metal exerting an important influence on the temperature of the central portion of the piston head.

The chief conclusion to be drawn from this review of heat flow in the steady state through thermal circuits containing metallic sections is that although there are unquestionably cases in which the thermal conductivity of the metal-

lic portions is significant, in most instances the conductivity of the metal is not a factor in controlling the total heat flow. The importance of the conductivity of the metal has been greatly exaggerated and over-emphasized and it is not uncommon to find an engineer concerned over some utterly insignificant difference in conductivity between two metals. As a matter of fact, if one were concerned only with heat flow in the steady state, it would be quite possible to get along fairly well with only the roughest ideas of the thermal conductivity of metals, or even with no knowledge of it at all. This does not apply, however, to heat flow in the unsteady state.

Bibliography—Chapter III

1. Jacobs and Starr, *Review of Scientific Instruments*, Vol. 10, 1939, p. 140.
2. W. Rädeker and Schöne, See letter from H. Hougardy, METAL PROGRESS, Vol. 39, 1941, p. 70.

Chapter IV

HEAT FLOW IN THE UNSTEADY STATE

Heat flow in the unsteady state, that is, heat flow in which the temperature is variable, is very common in thermal circuits which contain metals; it occurs whenever a mass of metal is heated or cooled and is therefore encountered in such processes as casting, the solidification of ingots, heating of slabs or billets, annealing, quenching and welding. In the steady state, which was considered in Chapter III, the primary interest is in the rate of transmission of heat through the metal but in the unsteady state the emphasis is rather on the rate of change of temperature within the metal. Yet transfer of heat through a metal and temperature change within it are not entirely independent, for the rate of change of temperature is proportional to the rate at which heat flows through the metal, and this rate is governed by the thermal conductivity of the material; the higher the thermal conductivity, the faster is the change of temperature, other things being equal. But rate of heat flow is not the only factor involved, since the rate of change of temperature depends also on the specific heat per unit volume for the metal in question, and, in fact, varies inversely with the volume specific heat; that is, the higher the volume specific heat, the smaller is the change in temperature produced by the addition or extraction of a given quantity of heat. The combination of these two factors gives a coefficient which characterizes the rate of change of temperature within a homogeneous material under a given set of conditions; it is defined as the ratio of the thermal conductivity to the volume specific heat and is denoted by the symbol a.* This coefficient

*The symbols h^2 and K are also used but a is the symbol recommended by the A.S.A. and will be used throughout.

was called the thermometric conductivity by Clerk Maxwell because it can be considered as a criterion of the rate of conduction of temperature through a material, and a translation of this term is still commonly used in Germany. Lord Kelvin called it the thermal diffusivity to emphasize the fact that it represents the diffusion coefficient of temperature; for it is entirely permissible to think of temperature as diffusing into a body. Of these two terms, the name thermal diffusivity has been the more widely accepted in English-speaking countries and is the one in common use today. It may seem somewhat strange at first to consider temperature, which is only an abstraction, as something which diffuses, but this is no more of an anomaly than speaking of the flow of heat and has much to recommend it in that it represents a useful viewpoint insofar as practical treatment is concerned. For the equations representing the rate of change of temperature are identical with those which represent the rate of diffusion of one substance in another, and the thermal diffusivity has the same dimensions as any other diffusion coefficient, namely $cm^2/sec.$ or $in^2/sec.$

As a matter of experimental observation, the value of the thermal diffusivity is largely determined by the value of the thermal conductivity, since the volume specific heat of the common metals varies but little, as is shown in Table IX. Thus, although the conductivity of the metals shown varies by 50 to 1 and the specific heat per unit mass varies by 10 to 1, the volume specific heat varies by only 3 to 1. This constancy of the volume specific heat is significant and is not as widely recognized as it should be, since there is frequent confusion of the specific heat per unit volume with that per unit mass, which is the quantity commonly found in tables of physical constants. For example, Table IX shows that the specific heat of aluminum per unit mass is greater than that of the other common metals. The conclusion is often drawn, there-

Table IX
Thermal Constants of Various Metals

Metal	Thermal Conductivity Cal.Sec.$^{-1}$ Cm.$^{-1}$ °C^{-1}	Density Grams per cc	Mass (c) Specific Heat Cal. per gram	(ρc) Volume Specific Heat. Cal. per cc	Diffusivity Cm^2/sec.
Aluminum	0.485	2.70	0.214	0.577	0.84
Antimony	0.044	6.68	0.050	0.335	0.13
Bismuth	0.020	9.80	0.029	0.284	0.07
Cadmium	0.223	8.6	0.059	0.507	0.44
Copper	0.927	8.92	0.092	0.820	1.13
Gold	0.707	19.3	0.031	0.600	1.18
Iridium	0.141	22.4	0.032	0.720	0.20
Iron	0.148	7.86	0.107	0.840	0.18
Lead	0.084	11.34	0.030	0.340	0.25
Magnesium	0.270	1.74	0.245	0.440	0.84
Molybdenum	0.349	10.2	0.065	0.660	0.53
Nickel	0.140	8.90	0.105	0.930	0.15
Palladium	0.161	12.0	0.058	0.700	0.23
Platinum	0.166	21.45	0.032	0.690	0.24
Rhodium	0.214	12.5	0.058	0.720	0.30
Silver	1.00	10.5	0.056	0.588	1.70
Tantalum	0.130	16.6	0.035	0.580	0.224
Tin	0.157	7.31	0.054	0.395	0.398
Tungsten	0.382	19.3	0.033	0.637	0.600
Zinc	0.270	7.14	0.092	0.658	0.410

To change values of diffusivity from cm^2/sec. to
cm^2/min. multiply by 60
cm^2/hour multiply by 3600
in^2/sec. multiply by 0.155
in^2/min. multiply by 9.29

fore, that aluminum has a high heat storage capacity. This is true enough in terms of mass or weight but so far as heat flow is concerned, it is the specific heat per unit volume which counts, and in this respect aluminum has no marked advantage over the other common metals. Its heat capacity in terms of Btu. per pound may be large but there are so few pounds ‹ per cubic foot that a unit volume has approximately the same heat capacity as an equal volume of another metal.

Inasmuch as the value of the diffusivity is in effect fixed by that of the conductivity, the variation of diffusivity with composition, temperature, pressure, and other factors which influence it, is substantially the same as the corresponding variation of conductivity, hence the conclusions drawn regarding the variation in thermal conductivity in Chapter II apply in general to the diffusivity as well. The most useful of these are the conclusions regarding the influence of composition and physical constitution, which make it possible to predict with reasonable accuracy how the diffusivity will vary from metal

to metal. The variation of diffusivity with temperature is commonly neglected, except as one computes an average diffusivity for the temperature range considered. The introduction of a diffusivity varying with temperature makes the mathematical treatment difficult and unwieldy out of all proportion to the benefit gained, since calculations involving the unsteady state are not commonly made with an accuracy which justifies taking account of such second order effects.

Even neglecting such variations the quantitative treatment of the unsteady state is vastly more complex than that of the steady state. The necessary equations involve special functions, they are exceedingly difficult to set up except for heat flow in simple geometrical shapes such as the sphere, the cylinder and the slab, and they require that the rate of change of temperature at the surface of the body or the rate of heat transfer across the surface be known.

INFLUENCE OF SURFACE CONDITIONS

As in the case of the steady state, heat and temperature must be supplied to or extracted from the surface of the body and the conditions under which this is done have a profound effect on the rate of change of temperature within the body. In many cases, indeed, the rate of change of temperature within a body of high diffusivity, that is, of high potential ability to equalize in temperature rapidly, is controlled by the slower rate at which heat can be made available at the surface, or what is essentially the same thing, the rate at which the temperature of the surface changes. Here again, it is necessary to take account of the whole thermal circuit and not only the metallic portions of it, since the thermal diffusivity of the films at the surface of the metal may be very low compared with that of the metal. Examples of the influence of such boundary conditions are easily found. In water quenching, for example, the formation of an insulating

Table X

Emissivity or Adsorptivity of Metals
A Perfect Radiator or Absorber = 1

Aluminum		
	Polished Plate	0.040
	Rough Plate	0.055
	Oxidized at 1110 degrees Fahr.	0.11 to 0.19
Brass		
	Highly polished	0.030
	Rolled plate	0.06
	Rubbed with emery	0.20
	Oxidized at 1110 degrees Fahr.	0.60
Copper		
	Polished	0.020
	Oxidized at 1110 degrees Fahr.	0.57
Iron and Steel		
	Polished	0.13 to 0.4
	Freshly machined	0.44
	Oxidized at 1110 degrees Fahr.	0.79
	Rough steel plate with oxide	0.94
Nickel		
	Polished	0.05
	Oxidized at 1110 degrees Fahr.	0.40
Silver		
	Polished	0.03

film of steam at the surface of the metal may greatly retard the cooling of the metal; in the so-called quenching towers in which liquid passes the metal surface under forced convection, the conditions of fluid flow have a measurable influence; in the cooling of ingots, the shrinkage of the skin away from the mold wall introduces a virtual resistance to the change in temperature.

Another significant factor influencing the rate of heating or cooling is the emissivity—or absorptivity—of the surface of the metal, which characterizes the rate at which the surface absorbs or emits radiant energy. The emissivity is conveniently expressed in terms of the amount of energy absorbed or emitted referred to that absorbed or emitted by a black body, which is a perfect radiator and absorber. If the emissivity is high relative to that for a black body, then the surface absorbs a large proportion of the energy which reaches it and heating is likely to be rapid, whereas, if the emissivity is low, more heat is reflected and only a little is absorbed so that heating tends to be slow. The value of the emissivity of some of the common metals is given in Table X, from which

it is evident that the emissivity varies markedly with the condition of the surface. The emissivity of a polished metal is very low, that of a rough surface is high, and since nonmetallic materials are in general better absorbers and radiators than metals, the presence of oxide on the surface commonly results in a high surface emissivity. A somewhat
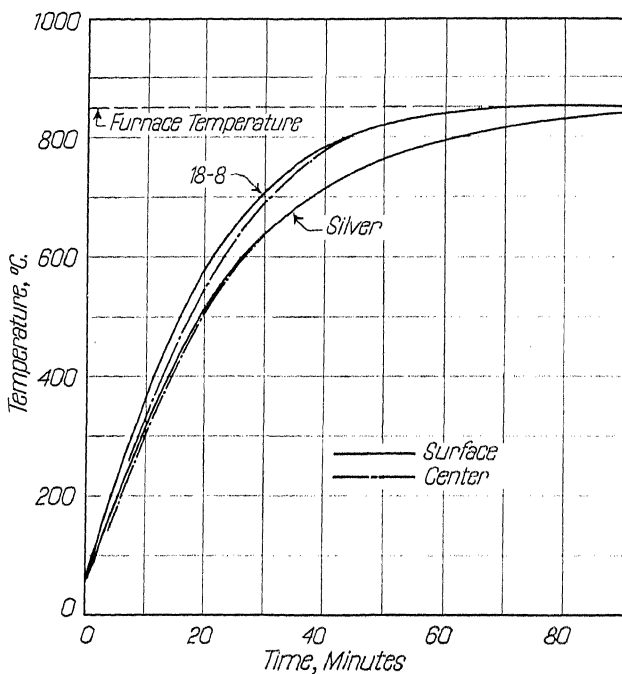
Fig. 44—The Rate of Change of Temperature in a Cylinder of Stainless Steel and a Similar One of Silver When Heated Under Identical Conditions.

extreme example of this influence of emissivity on the rate of heating is provided by the curves in Fig. 44, which shows the rate of change of temperature at the surface and center respectively of two cylinders, 3 inches in diameter and 7 inches long, one of bright silver, which has a very high thermal conductivity but a low emissivity, the other of oxidized 18-8 stainless alloy, which has a low conductivity but a high emis-

sivity. Both cylinders were originally at a uniform temperature of about 20 degrees Cent., and both were placed in a furnace maintained at 850 degrees Cent. (1560 degrees Fahr.). The temperature of the surface was determined by means of a thermocouple peened to the surface, that at the center by another inserted in an axial well. The results show that in spite of the fact that the thermal diffusivity of silver is some thirty times greater than that of the stainless alloy, the cylinder made of stainless alloy came to temperature more quickly than did the silver, because the bright silver surface reflected so much heat that the rate of transfer of heat to the cylinder was much less than the corresponding rate to the oxidized alloy which has a high absorptive capacity. The difference in thermal diffusivity was evident somewhat in the distribution of temperature within the metal, since the temperature difference between the surface and center of the silver cylinder was at no time as great as that in the stainless alloy, and in fact the temperature within the silver was virtually uniform above 600 degrees Cent., or long before the cylinder had come to equilibrium with the furnace.

The foregoing example is, admittedly, extreme, but similar effects of lesser magnitude are by no means hard to find. A typical example is illustrated in Fig. 45, taken from Stanfield, which shows the rate of heating at the center of a 2-inch cylinder of mild carbon steel heated from 20 to 1000 degrees Cent. (68 to 1830 degrees Fahr.) when (a) the surface is machined, that is, it is fairly bright so that it has a relatively low emissivity, and (b) when the surface is oxidized and has a relatively high emissivity. The data show that the higher rate of absorption of radiant energy by the oxidized surface results in appreciably faster heating at the center.

The influence of surface condition on the rate of heating within the metal is greatest for small or relatively thin pieces, for which the thermal diffusivity is relatively of little sig-

nificance.   As the dimensions of the piece became longer the relative importance of emissivity of the surface and of thermal diffusivity changes until for large · or thick masses, the diffusivity tends to be the controlling factor.   This is demonstrated by the typical example in Fig. 46, taken from Stanfield.
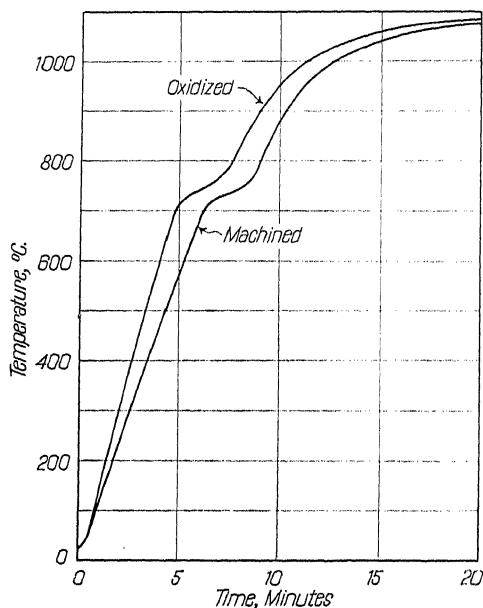
Fig. 45—The Influence of Surface Conditions Upon the Rate of Heating at the Center of a 2-Inch Cylinder of Steel. (After Stanfield.)

The two sets of curves in Fig. 46 are calculated for two sets of cylinders, one 2 inches in diameter, the other two feet across.   In each set, one cylinder has a high conductivity and diffusivity, about equivalent to that of low-metalloid iron, the other has a much lower conductivity and diffusivity, being in fact about that of 18-8 stainless alloy.

It was postulated that the cylinders were originally at a uniform temperature of 0 degree Cent. and that they were placed in a furnace maintained at 1000 degrees Cent. (1830 degrees Fahr.).   It will be observed that the rate of heating

at the center of the 2-inch cylinders was virtually identical in spite of the difference in diffusivity, indicating that the controlling factor was not the rate of equalization of temperature within the cylinders but was the rate at which heat was supplied to the surface. On the other hand, in the 2-foot
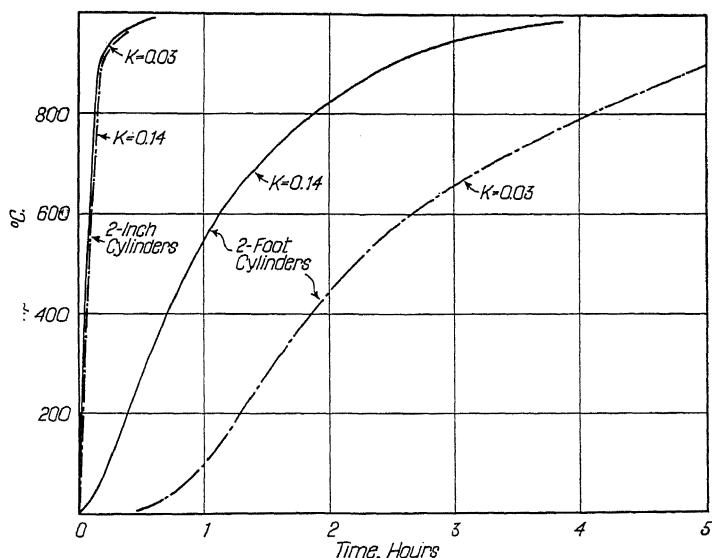
Fig. 46—The Rate of Heating at the Center of Two Sets of Cylinders of Different Thermal Conductivity, One Set Having a Diameter of Two Inches, the Other of Two Feet. (After Stanfield.)

cylinders there is a marked difference between the two because in this case the slow process, and therefore the rate controlling process, is the flow of heat within the cylinder and not, as before, the supply of heat to the surface. It is obvious from this example, that some caution must be used in deducing the behavior of a large mass from observations on a small piece.

Although the preceding examples are all cases of heating, the conclusions drawn are equally applicable to cooling, especially quenching. For a small mass of metal, the effectiveness of the quench depends almost entirely upon the rate at which

heat is extracted from the surface and is largely though not entirely, independent of the properties of the metal. In quenching a large mass, the properties of the metal become of much greater significance.

In view of the many factors which influence the rate of change of temperature at the surface, it is generally impossible to foretell it, to measure it, or even to represent it satisfactorily in a simple way. In solving problems of the unsteady state it is necessary therefore to make a series of compromises and assumptions. It must be assumed that the body has a simple geometric shape, such as a slab or a cylinder, recalescence effects must be neglected and some relatively simple postulate must be made regarding the rate of heat transfer at the surface or the rate at which the surface temperature changes. In treating the rate of heat transfer at the surface it is usual to assume Newton's Law of Heating or Cooling which states that the rate of heat transfer is proportional to the difference in temperature between the surface of the body and its surroundings; if the rate of change of surface temperature is to be used, it is common to assume that it changes instantaneously from one temperature to another, which corresponds to an infinite rate of heat transfer at the surface, or that it varies linearly with time. Considering the number of these assumptions which must be made, the results are of necessity only approximate, yet they serve admirably on a surprizingly large number of problems, one excellent example being the treatment of quenching made by Grossmann, Asimow and Urban.

Since the mathematics involved in treating the unsteady state is so difficult and since a knowledge of the basic equations is not necessary in using the results, a number of attempts have been made to represent the results in a convenient and nonmathematical form, usually as graphs or tables. Good examples of the graph type of treatment are those given by

Schack and by Bachmann, whereas the table type has been successfully employed by Newman and by Russell. The table type has the advantage that from it a graph can be constructed on any desired scale and it is therefore often possible to make more precise calculations than is possible from the relatively small graphs usually available in publications.

Russell's Tables are particularly noteworthy because they are unusually precise, being given to five places of decimals, and because they cover a wide range of conditions. They represent a tremendous amount of laborious and tedious calculation and they deserve much wider recognition than they seem to have achieved as yet. Since such Tables, or their equivalent, are essential in studying problems of heat flow in the unsteady state, Russell's Tables are reproduced in Appendix I.

<div align="center">THE HEAT FLOW TABLES</div>

The particular assumptions made by Russell were: (a) The diffusivity is independent of temperature; (b) Newton's Law of Heating or Cooling is obeyed; (c) The body is initially at uniform temperature; (d) There are no recalescence or other anomalous heat affects.

In order to present the results in such condensed form, relative rather than absolute variables are used, and these can be reduced to the following four:

(1) *A relative temperature*—Since the basic equations involve only differences in temperature and not temperature directly, it is convenient to use a relative temperature, U, which is the fraction of the total temperature change ultimately to be attained which remains to be accomplished at any given instant. Thus,

$$U = \frac{\text{Temperature at a given time—Final uniform temperature}}{\text{Initial uniform temperature—Final uniform temperature}}$$

For example, at the start, all the temperature change remains

to be accomplished, hence U is 1, whereas when temperature equilibrium is attained it is zero. If a piece of metal initially at 0 degree Cent. is placed in a furnace maintained at 1000 degrees Cent., then when the temperature at a given point in the piece is 800 degrees Cent., the value of U is $\dfrac{800-1000}{0\text{-}1000}$ or 0.20.

(2) *A relative position*—This generalized variable ex-
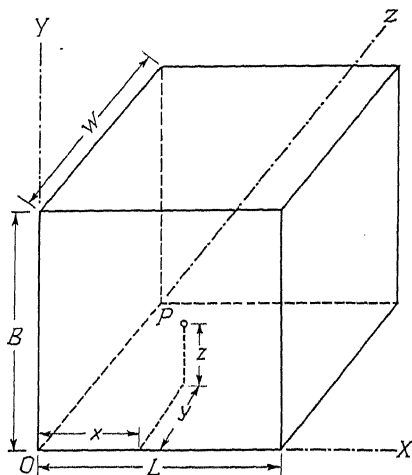
Fig. 47—Illustration of the Method of Determining the Relative Position of a Point P Within a Rectangular Block of Dimensions 2L x 2B x 2W. Origin of Coordinates at Center of Block.

presses the location of any point within a body in terms of its relation to the center or central axis of the piece and is given numerically by the ratio of the distance of the location considered from the center to the total distance between surface and center. For example, in a cylinder of radius 2.0, a point which is 1.8 units from the axis has a position ratio of 1.8/2.0 or 0.9. Or in a rectangular block such as that shown in Fig. 47, if the dimensions of the block are 2L×2B×2W, then the position of point P, which is x units from the center in the X-direction, y units in the Y-direction, and z units in

the Z-direction, is given by the three ratios x/L, y/B, z/W. In this way the absolute dimensions of the body are eliminated.

(3) *A relative time ratio which we shall designate* τ — Thus, instead of using time directly it is convenient to use the ratio $\tau = at/L^2$, where a is the thermal diffusivity, t is the elapsed time measured from the beginning of the change in temperature, and L is the radius or half-thickness of the piece, that is, it is the distance between the center or axis and the surface. This generalized relative time ratio takes into account the difference in diffusivity and in size of different pieces. Those familiar with the laws of diffusion will recognize that this is the same ratio used in problems of diffusion.

(4) *A factor representing the heat transfer across the surface*—Newton's Law of Heating or Cooling may be written

$$q = S \ (T_1 - T_2) \tag{20}$$

where q is the rate of heat transfer per unit surface area, T_1 and T_2 are the temperature of the surroundings and metal surface respectively, and S is a proportionality constant. Now in solving the heat flow equations it is convenient to use a factor which takes into account the rate of heat transfer at the surface and also the properties of the body in question. To do this, a coefficient of surface transfer, H, is employed. This coefficient is numerically equal to S/K, where S is the constant in equation (20) and K is the thermal conductivity. The rate of heat transfer across the surface of a given metal when heated in a particular furnace or cooled in a particular quenching bath can therefore be characterized by a specific value of the coefficient H.

In addition to this surface coefficient, however, it is necessary to take account of the dimensions of the body being heated or cooled and this is done by multiplying H by L, the half-thickness or radius of the piece. The product HL gives therefore a factor which characterizes the rate of heat trans-

fer at the surface of a specific piece of metal under specific conditions of heating or cooling.

The Tables given in the Appendix are made up for different values of the factor HL; the value of U, the relative temperature, being then given as a function of τ, the relative time for several relative positions.

It should be noted that all these relative variables are dimensionless, hence, any set of units may be employed so long as they are consistent within the particular ratio being used. For example, in the position ratio it makes no difference whether inches or centimeters are used so long as the same unit is employed for both distances involved in the ratio. Moreover, it is not necessary to use the same system of units for all the relative variables, for example, inches can be used for the position ratio and at the same time the relative temperature can be computed using Centigrade temperature, if that is convenient.

The Tables given in Appendix I apply to heat flow in an infinite plate or slab and in an infinite cylinder, in both of which shapes the heat flow is unidimensional and perpendicular to the infinite surface. These are the fundamental cases and a surprisingly large number of instances in practice can be considered as close approximations to them. For example, any slab whose length and breadth are more than about 5 times the half-thickness can be treated as an infinite slab without appreciable error; or any cylinder whose length is more than 5 times its radius is for all practical purposes an infinite cylinder; or a long rectangular rod can be roughly considered an infinite cylinder.

As an illustration of the use of the Tables, suppose that there is a slab of steel, 105 inches long, 40 inches wide, and 6 inches thick, which is initially at a uniform temperature of 70 degrees Fahr. and which is to be heated by passing it through a furnace maintained at 2300 degrees Fahr. (1260

degrees Cent.). How long will it take for the center to come to 2000 degrees Fahr. (1090 degrees Cent.), and what is the surface temperature when the center reaches 2000 degrees Fahr. (1090 degrees Cent.)? When the center temperature is 2000 degrees Fahr. (1090 degrees Cent.), the value of U is $\frac{2000 - 2300}{70 - 2300} = 0.135$. The diffusivity, a, of steel can be taken as 1.00 in²/min and from previous experience it is known that H, the surface coefficient is 0.1, hence the value of HL is $0.1 \times 3 = 0.3$. Since we are concerned first with the temperature at the center, the value of the position ratio, x/L, is zero. Now there is no Table for HL = 0.3, so that the corresponding values of U and τ must be obtained by interpolation between the Tables for HL = 0.2 and HL = 0.5. This can be done graphically as shown in Fig. 48. Draw the curve for U at x/L = 0 as a function of τ for HL = 0.2 and 0.5. Then draw a horizontal line at U = 0.135 and on this line estimate the position of the curve for HL = 0.3 as one-third of the distance between the two curves. The value of τ corresponding to this position is 9.0, that is,

$$\frac{a\,t}{L^2} = 9, \text{ or } t = \frac{9 \times 3^2}{1.00} = 81 \text{ minutes}$$

The center of the slab therefore reaches 2000 degrees Fahr. (1090 degrees Cent.) after 81 minutes.

The surface temperature at this time may also be calculated. Draw a pair of curves for x/L = 1, similar to those shown in Fig. 48 for x/L = 0, then estimate the position of the curve for HL = 0.3. The value of U on this curve for τ = 9.0 is about 0.12, hence,

$$\frac{T - 2300}{70 - 2300} = 0.12$$

and T = 2033 degrees Fahr (1110 degrees Cent.). In other words under these conditions, there is only 33 degrees difference between center and surface temperature.

As a general rule in heating masses of metal of not too great bulk under most conditions the center temperature does not lag far behind that of the surface and as the temperature of the surface approaches the equilibrium temperature the temperature gradient is very small. This fact is evident in Fig. 44. The same is true on cooling, if it is relatively slow, but it does not apply, of course, to quenching.
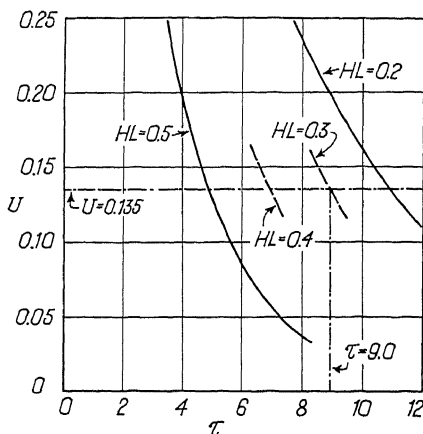
Fig. 48—Illustration of the Method of Finding a Value of the Relative Time, τ, by Graphical Interpolation.

In a few cases, as for example, heat flow in a cylinder whose length equals its diameter, or in a rectangular block, the Tables do not apply directly but can still be used, as is illustrated below for the most difficult case, that of the temperature change at a point P, within a rectangular block of dimensions $2L \times 2B \times 2W$, such as is illustrated schematically in Fig. 47.

To take a specific case, it is desired to calculate the rate of cooling at the center of a pack of steel sheets, 150 inches long, 66 inches wide and 40 inches thick which has been heated in an annealing cover to a substantially uniform temperature of 2300 degrees Fahr. (1260 degrees Cent.) when the an-

nealing cover is removed. It has been established by previous work that under the conditions existing on such cooling, the value of H, the surface transfer coefficient, is 0.05; it has also been determined that the effective diffusivity of stacked steel sheets is 10 in²/hr or 0.166 in²/min. This is, as a matter of fact, a value actually determined in one case and it is very
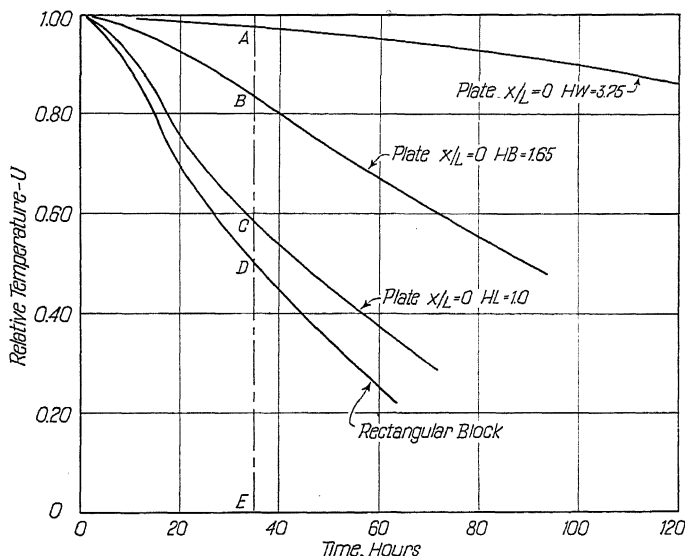
Fig. 49—Illustration of the Method of Finding the Rate of Change of Temperature Within a Rectangular Block.

much less than the value of 60 in²/hr or 1.0 in²/min which holds for a solid block. The difference is due to the many thermal resistances provided by the contacts between the sheets.

The general method is to calculate the cooling curve for the point P as though it were in three infinite plates of thickness 2L, 2B, and 2W, that is, 40 in, 66 in, and 150 inches.

$$\text{Since } H = 0.05, \ HL = 1.00$$
$$HB = 1.65$$
$$HW = 3.75$$

and since the point P is at the center x/L, y/B and z/W are
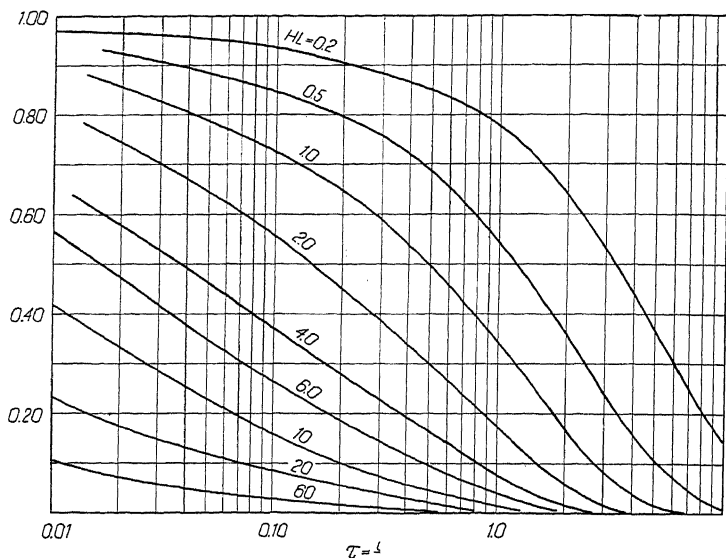
Fig. 50—The Variation of Relative Temperature U at the Surface of a Slab with Relative Time τ for Different Values of the Surface Transfer Coefficient HL.

all zero. The time of removing the annealing cover is taken as zero.

Now from the Tables draw the cooling curve for the center, $x/L = 0$, for an infinite plate when $HL = 1.0$, only make the abscissa time instead of τ. This is readily done since $\tau = \frac{a}{L^2} t$, and for a 40-inch plate of steel, $t = \frac{400}{10} \tau$. On the same graph draw similar curves for $HB = 1.65$ and $HW = 3.75$ obtained by interpolation in the Tables. These are curves B and A respectively in Fig. 49. Now read off the value of U on each curve for a given value of time and multiply these three values to obtain the value of U for the pack of sheets. Thus, in Fig. 49, $DE = CE \times BE \times AE$. By repeating the process at other values of time the whole curve for the block can be obtained. Values of U can then be calculated into temperature to give the cooling curve.
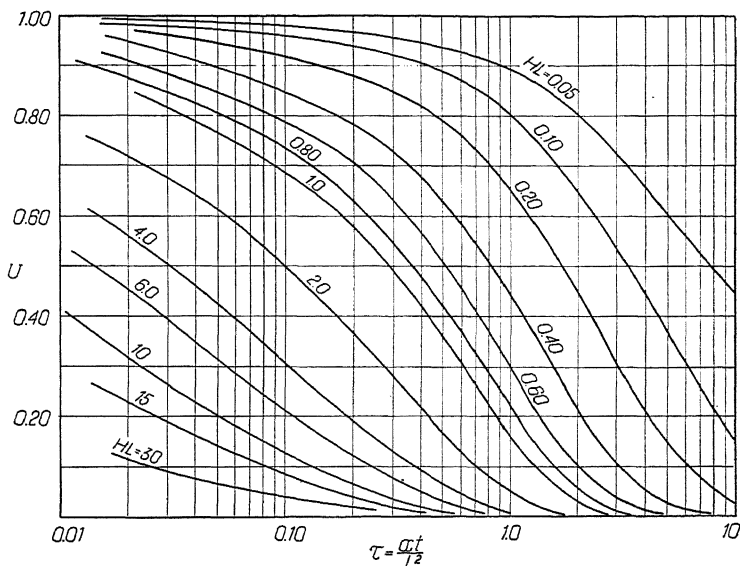
Fig. 51—The Variation of Relative Temperature U at the Surface of a Cylinder with Relative Time τ for Different Values of the Surface Transfer Coefficient HL.

DETERMINATION OF SURFACE COEFFICIENT, HL

It has been assumed thus far that the value of H, the heat transfer coefficient, is known. This is not always the case, so that it is necessary to discuss methods of determining it. In principle, H can be determined if the diffusivity is known and if the rate of change of temperature at any given point within the metal can be determined, since one can then plot a curve for U versus τ for the given position and can match this curve against a set plotted from the Tables. And once determined for a given metal and a given furnace condition or given cooling conditions, the values can be used for other pieces of the same metal. In practice, it is commonly most convenient to measure the rate of change of temperature at the surface and when this has been determined the curve for U against τ can be matched against the curves plotted from the data in the Tables for various values of H at x/L = 1.
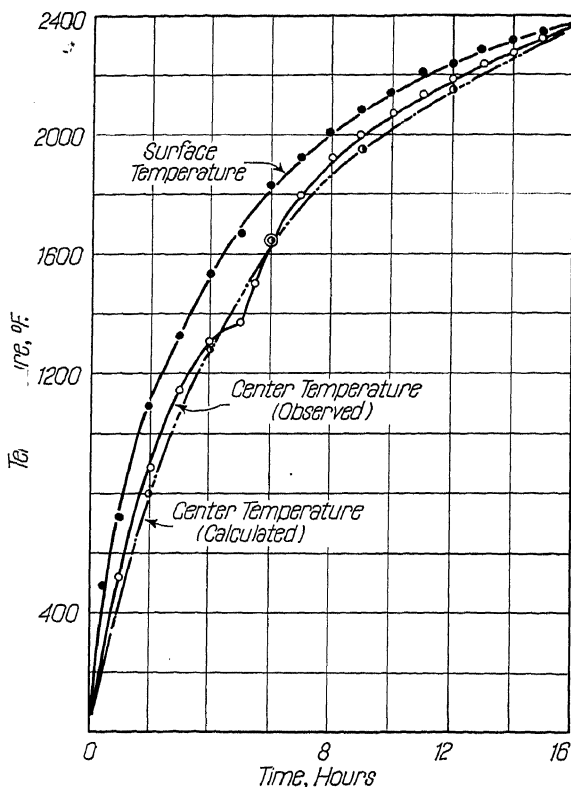
Fig. 52—The Rate of Heating at Surface and Center
of a 19-Inch Ingot of S.A.E. 1045 Steel Placed in a
Cold Soaking Pit and Heated to 1500 Degrees Fahr. (815
degrees Cent.). (Carnegie-Illinois Steel Corporation.)

Such graphs for the slab and cylinder are shown in Figs. 50
and 51 respectively.

The use of this method is illustrated by a typical example,
which also serves to show how closely certain shapes approach
the ideal ones for which the Tables were calculated.   An ingot
19 inches square and 60 inches high of S.A.E. 1045 steel was
placed cold in a relatively cold soaking pit to be heated to
2300 degrees Fahr. (1260 degrees Cent.) in about 15 hours.
The surface temperature was measured with a thermocouple
with results shown by the black disks in Fig. 52.   For this

steel, $a = 1.0$ in²/min. Values of $\dfrac{a\,t}{L^2} = \dfrac{1.0}{(9.5)^2}$ t and U were calculated from these observations and plotted on a graph of the type shown in Fig. 53, drawn from the data for an infinite cylinder. It is evident that HL has a value of about 0.15,

hence H $\dfrac{0.15}{9.5} = 0.016$. This value can therefore be re-
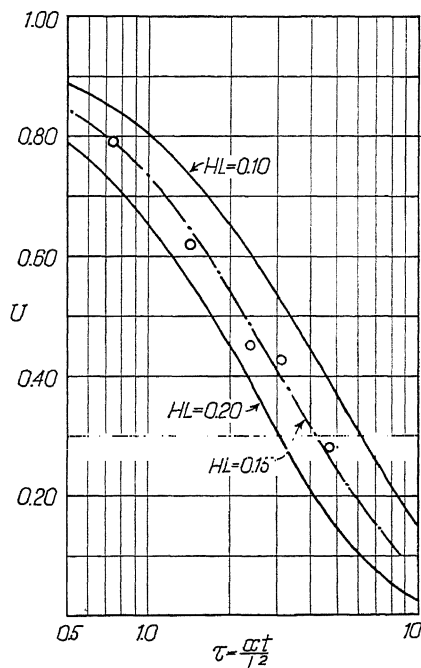
Fig. 53—Illustration of the Method of Determining the Value of the Coefficient HL from Data on the Rate of Change of Temperature at the Surface of a Cylinder.

garded as typical for heating ingots in this furnace under these conditions.

It should be noted that in Fig. 53, the form of the experimental curve closely matches those taken from the Tables, which indicates that Newton's Law was substantially obeyed in this case.

It is also of interest to calculate the rate of heating at the center to see how closely this ingot shape approaches a cylinder. The value of HL is 0.15, hence by interpolation between the Tables for HL 0.1 and 0.2 it is possible to construct the curve for U at x/L = 0 as a function of τ. This has been done with results shown by the dotted curve in Fig. 52. This curve agrees fairly well with the measurements made by means of a thermocouple inserted to the center in an axial well, the chief discrepancy being caused in part, at least, by the absorption of heat accompanying the phase change at 1400 degrees Fahr., an effect which is neglected in the calculations. These data indicate, therefore, that an ingot of this size and shape behaves substantially as if it were an infinite cylinder; they also demonstrate the extent to which neglect of the recalescence effects influences the calculated values.

In many cases it is not possible to determine the rate of change of surface temperature, and this is particularly true of quenching. If the metal being quenched has a characteristic structure, as is the case with steels, there is available an excellent method devised by Grossmann, Asimow and Urban which is based on the axiom that in quenching a set of cylinders, every point in the several cylinders which has the same structure after quenching had the same rate of cooling. By comparing the structure in cylinders of several sizes it is possible to eliminate the actual rate of cooling, and to make the problem in effect one of finding what value of H will give the same rate of cooling at the positions selected in each cylinder as having the same structure. To do this these authors have derived a set of curves for the temperature change in cylinders which is substantially identical with Russell's Tables given in Appendix I. For the details of this method the reader is referred to the original publications but the essence of the scheme is the chart shown in Fig. 54.

Cooling Power of Quenching Baths

Curves whereby severity of quench (heat transfer equivalent H) can be estimated. Copyright, 1939, Carnegie-Illinois Steel Corp.

$\dfrac{D_u}{D}$ (vertical axis)

D or HD (horizontal axis)
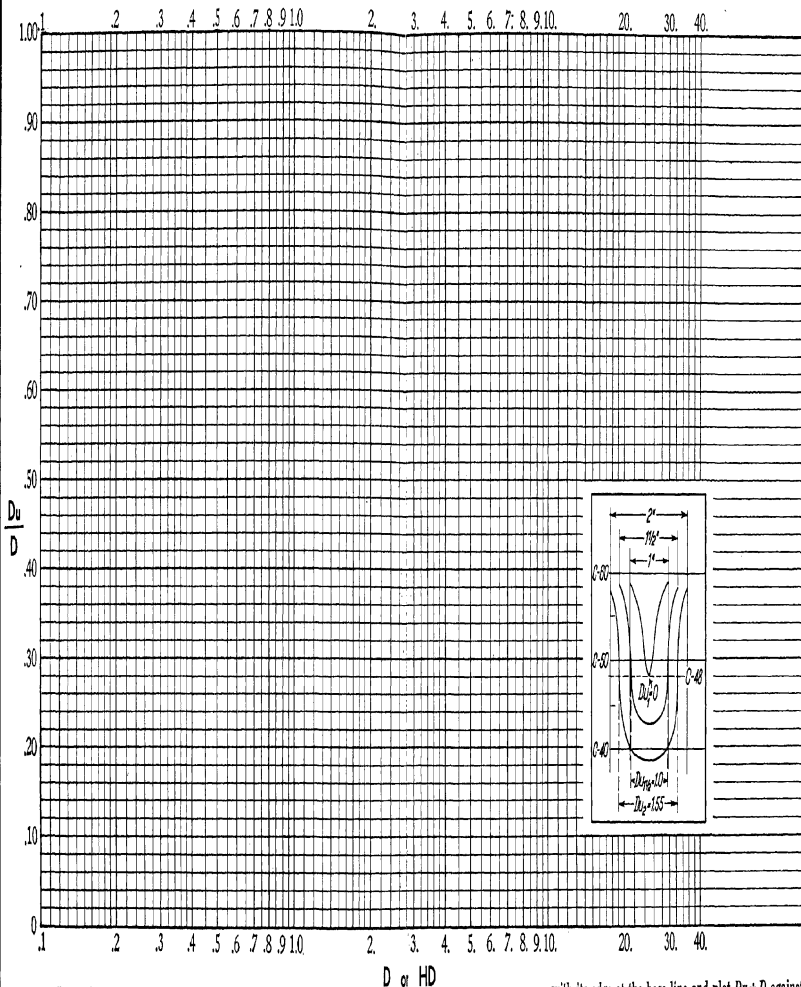
By M. Asimow and M. A. Grossmann

TO USE this chart, harden *at least* three bars of various diameters in the given medium at working temperature and circulation, and determine curves of hardness distribution, edge to axis, at midlength. In the example sketched in the inset above, D is 1, 1.5 and 2 in. respectively. At a selected hardness horizontal, as C-48, which intersects or touches all curves, determine the ratio $D_u \div D$ between soft center and bar diameter. (Soft center diameter D_u is 0, 1.0 and 1.55 in.; $D_u \div D$ is 0, 0.66 and 0.77 respectively.) Lay a piece of tissue paper over the chart with its edge at the base line and plot $D_u \div D$ against D for each bar size. Slide tissue paper right or left until these points fall on a single curve, which for this quench happens when D_l is at 5.0 and $D_{1.5}$ is at 7.5 (these now being values of HD). Then H, the severity of this quenching set-up, is $HD \div D$ or $5.0 \div 1.0$ or 5.0 (or $7.5 \div 1.5$ which gives the same result).

Fig. 54

This chart expresses the severity of quench in terms of the surface coefficient H.

To determine the severity of quench, that is, the surface coefficient, H, select a single bar of steel and prepare from it a series of at least three rounds of different diameter. The
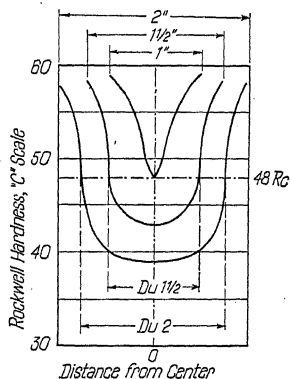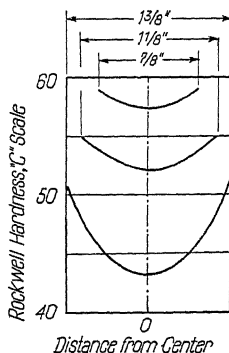
Fig. 55a   Fig. 55b

Fig. 55—(a) Hardness Distribution Suitable for Determining the Value of the Coefficient HL, That Is, the Severity of Quench. (b) Hardness Distribution Unsuitable for Determining the Severity of Quench.

bar size should be adjusted to the hardenability of the metal so that after quenching the hardness range of the several cylinders overlaps, thus making it possible to select one hardness which occurs in all three cylinders. For instance the curve shown in Fig. 55a is suitable whereas that shown in Fig. 55b is not suitable.

As an illustration of the method, take the curves in Fig. 55a. It is now necessary to determine the ratio of the unhardened diameter, Du, to the diameter of the bar, D. In order to measure Du, select a hardness which is common to all three bars, such as Rockwell 48 C. Now on the 1½ inch bar, the diameter bounded by the hardness Rockwell 48 C is indicated by the distance marked $Du_{1\frac{1}{2}}$ in Fig. 55, and corresponds to 1.00 inch; hence Du = 1.00, and the ratio Du/D = 1.00/1.50 = 0.66. The unhardened diameter of the 2-inch

bar, marked Du_2 in Fig. 55, is 1.55 inches, hence, the ratio
Du/D is 1.55/2.00 = 0.77. The curve for the 1-inch bar just
touches the position Rockwell 48 C, so that the value of Du
and likewise of Du/D is zero. We therefore have the follow-
ing results:

D	Du/D
1.00	0.0
1.50	0.66
2.00	0.77

These points are now plotted on the black co-ordinates of
Fig. 54 by placing a piece of transparent paper, tracing cloth
or celluloid over the chart, being careful to place the trans-
parent material so that its lower edge coincides with the lower
edge of the paper or of the chart. Thus, on the bottom line
find the position 1.0 on the abscissa, which is the D or HD
scale, and plot the first point on the bottom line. This is the
point at which Du/D is zero. Then find the position 1.50 on
the horizontal axis and follow the black vertical line at this
position upward until it meets the horizontal line for Du/D =
0.66, which fixes the position of the second point. Plot the
third point similarly at the intersection of the black lines for
D = 2 and Du/D = 0.77. These three points are now to be
matched against the orange curves. Slide the transparent
cover over the chart being careful to keep it properly aligned,
until the three points fall on the same orange curve. This
happens when the bottom point is at position 5.0, the inter-
mediate point on the black vertical line for 7.5 and the third
point on the vertical black line for 10.0. The orange curves
are plotted for HD, hence the orange curve matched shows
that for the 1-inch bar (the lowest point) HD = 5.0, whence
H = 5.0/D = 5/1.0 = 5.0. Similarly, for the 1.5-inch bar,
HD = 7.5, whence H = 7.5/1.5 = 5.0 and for the 2-inch bar
HD = 10.0 or H = 10.0/2.0 = 5.0. The severity of quench or
heat transfer equivalent is therefore 5.0. This method has
proved very useful in practice and is now being refined to the

Table XI
Typical Values of the Surface Coefficient H for the Heating or Cooling of Steel

Heating	H in in.$^{-1}$
Sheets heated in a radiant tube furnace	0.05
Slab heating furnace	0.10
Soaking Pit	0.025
Electrical resistance (Globar) furnace	0.03
Cooling	
Still air	0.02
Air Natural convection	0.05
Still Oil*	0.30
Still water*	1.0

*The value of the surface coefficient for moving oil, or water is appreciably greater than that for the same coolant when still but it varies so much with the type of motion relative to the metal that any single value may be misleading, hence, none is given.

point where deviations from the basic assumptions are becoming detectable.

One of these basic assumptions is that the coefficient HL is independent of the size of bar used. This is substantially valid if the variation in size does not alter the conditions of quenching, as is the case if the bar is quenched into a large bath, but it may not be valid if a change in bar diameter alters the rate of flow past the surface, as it may in certain types of apparatus, such as the so-called quenching tower. Care must always be taken therefore to insure that boundary conditions remain constant when the bar diameter is changed.

Typical values of the coefficient H for heating or cooling steel under various conditions are shown in Table XI. It will be observed that the values for heating show a relatively small variation, the range being about 5 to 1, whereas those for cooling and quenching show a much larger spread, the variation being about 50 to 1.

Should there be no method available for determining the coefficient, H, the maximum rate at which a piece may be heated or cooled, or the minimum time required for a piece to reach temperature equilibrium can be calculated from the Table for HL = ∞, which corresponds to an instantaneous change in surface temperature from one constant value to another. It should be noted that with such an instantaneous change of surface temperature the product HL is infinite so

that this single Table applies to pieces of all sizes. More-
over, for a given value of U, there is a unique value of τ
for a given relative position. Since $\tau = \dfrac{a\,t}{L^2}$, it follows directly
that the time required to reach the given value of U at the
given relative position varies inversely as the square of the
radius or half-thickness. For example, the time required to
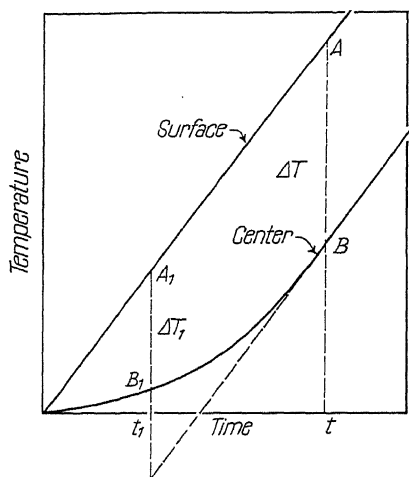
Fig. 56—Rate of Change of Tempera-
ture at the Center of a Body When the Sur-
face Temperature Increases Linearly with
Time. (After Russell.)

bring the center of a cylinder half-way to the equilibrium
temperature (U = 0.50) is 4 times as great for a bar 2 inches
in diameter as for a bar 1 inch in diameter. This relation,
though not strictly valid for other values of H, does, never-
theless, provide a useful approximation of the effect of vary-
ing size on the time required for heating or cooling.

HEAT FLOW WHEN SURFACE TEMPERATURE CHANGES AT A CONSTANT RATE

Occasionally, one encounters a type of heating or cooling
in which the surface temperature changes at a constant rate,

Table XII
Maximum Temperature Difference Δ T Between Surface and Center of Various Shapes When Surface is Heated at a Uniform Rate of Θ Per Minute

Shape		ΔT
Infinite Plate of Thickness 2L—Heated from Both Sides		$\dfrac{\Theta L^2}{2a^2}$
Infinite Cylinder of Radius b		$\dfrac{\Theta b^2}{4a^2}$
Finite Cylinder, Diameter 2b = Length 2L	0.201	$\dfrac{\Theta L^2}{a^2}$
Infinite Rectangular Rod 2L × 2W, if W = L	0.295	$\dfrac{\Theta L^2}{a^2}$
if W = 2L	0.455	$\dfrac{\Theta L^2}{a^2}$
Cube 2L on a Side	0.221	ΘL^2

that is, the surface temperature varies linearly with time according to the equation

$$T = \Theta t$$

where T is the temperature at time t, and Θ is the constant rate. Such a rate of change is sometimes observed in furnace heating and is occasionally met in air cooling. This type of change has also been considered by Russell with concise and useful results. The most significant conclusion is that after an initial period which is relatively short, all parts of the body increase in temperature at the same rate as the surface, as is illustrated in Fig. 56. Moreover, if ΔT is the maximum difference in temperature between the surface and center of a body heated under these conditions, that is, if ΔT is the constant difference after the linear rate of change of temperature has been established, see Fig. 56, then ΔT can be expressed in terms of Θ and the dimensions of the body as shown in Table XII.

If ΔT_1 be the temperature difference between surface and center at a time t_1 before the maximum difference is established, see Fig. 56, then the value of $\dfrac{\Delta T - \Delta T_1}{\Delta T}$, that is, the

fractional part of the temperature difference still to be achieved, is given for various bodies as a function of τ by the curves in Fig. 57.

The foregoing discussion and examples illustrate the application of the fundamental principles describing the flow of heat to the basic processes of heating or cooling a solid
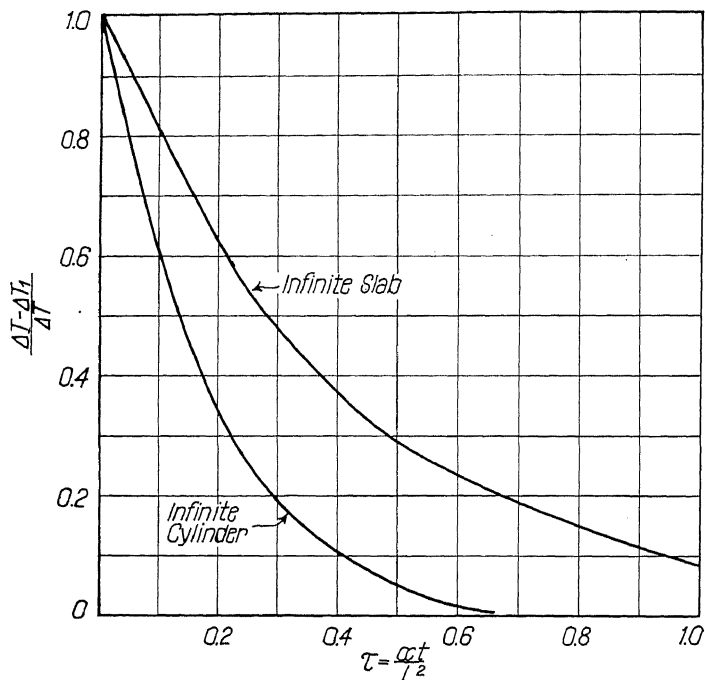
Fig. 57—The Variation of $(\triangle T - \triangle T_1)/\triangle T$ with τ for the Infinite Slab and Cylinder When the Surface Temperature Increases Linearly with Time. (After Russell.)

body of metal of relatively simple shape, such as an ingot, a slab or a billet, and they provide an introduction to the special problems connected with quenching. There are, of course, other useful applications of these principles, notably to the solidification of ingots and to welding or cutting, but consideration of such specialized topics has not been possible within the limited compass of these lectures. For the details

of these subjects the interested reader is referred to the many articles in the literature. To mention but a few, the papers by Asimow and Grossmann and their colleagues cited in the Bibliography for this Chapter provide a most instructive and comprehensive discussion of the heat flow in quenching; the solidification of ingots has been considered by a number of authors, notably, Chipman and FonDersmith, Roth, Heggie, McCance, Russell (2), and Matuschka; a fairly comprehensive mathematical treatment of heat flow in welding and cutting, together with a bibliography of significant papers in this field, has been given by Rosenthal, to supplement which there is a more popular account of thermal effects in welding by Aborn. These several articles, and indeed most of the contemporary papers on heat flow, are somewhat advanced, but the reader should not let this fact dissuade him from studying ꞏhem, since, in general, it is possible to understand their conclusions without following the detailed methods by which they were derived.

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12. G. Stanfield, First Report Alloy Steels Research Committee, Iron and Steel Institute Special Report, Vol. 14, 1936, p. 129.

These Tables, taken from Russell (First Report of the Alloy Steels Research Committee, Special Report 24, of the Iron and Steel Institute 1936) are used to calculate the rate of change of temperature or the temperature distribution within an infinite plate or slab and within an infinite cylinder or rod. The data are presented in terms of four generalized variables:

(1) A surface coefficient, HL, which characterizes the rate of heat transfer across the surface of the body. It is composed of two factors: L, which is the radius or half-thickness of the body, and H, a coefficient which represents, in effect, the rate at which heat is supplied to the surface of the rod or slab. The value of H depends on the particular metal under consideration and on the particular type of heating or cooling used, but once determined for a given combination of metal and method of heating or cooling, the value can be used for all sizes or shapes. For example, in heating slabs in a particular furnace, H was found to be 0.10 in^{-1}, hence, for a 6-inch slab HL $= 0.10 \times 3 = 0.30$.

(2) A relative temperature, U, which is the fraction of the total temperature ultimately to be achieved which remains to be accomplished at a given instant. Thus,

$$U = \frac{\text{Temperature at given time—Final uniform Temperature}}{\text{Initial uniform Temperature—Final uniform Temperature}}$$

Hence, at the start of heating or cooling, when all the temperature change remains to be accomplished, U is unity whereas at final equilibrium U is zero.

(3) A relative position (x/L) which expresses the location of any point within a body in terms of its relation to the center or central axis of the piece and is given numerically by the ratio of the distance of the point from the center (x) to the total distance between surface and center (L). For

example, on the central axis of a cylinder x/L ⋅ 0, at a point midway between the axis and the surface x/L = 0.50, and at the surface x/L = 1.

(4) A relative time ratio, τ, which is equal to $\frac{a\,t}{L^2}$, where a is the thermal diffusivity of the material, L is the radius or half-thickness, and t is the elapsed time since the beginning of heating or cooling. This time ratio therefore takes account of differences in size and in diffusivity.

As an illustration of the use of the Tables suppose that a steel cylinder 4 inches in diameter and 4 feet long, initially at 70 degrees Fahr. is to be heated in a furnace held at 2200 degrees Fahr. (1205 degrees Cent.), the diffusivity of the steel is 1.0 in²/min, and H has a value of 0.05 in⁻¹. How long will it take the center of the rod to come to 2000 degrees Fahr. (1095 degrees Cent.)? The coefficient HL has a value of $0.05 \times 2 = 0.10$, hence, we use the Table headed HL = 0.10. The value of U is $\frac{2000 - 2200}{70 - 2200} = 0.094$, and since we are interested in the center, x/L = 0. Now looking down the column for x/L = 0, it is found that when U = 0.094, τ has a value of about 12, hence $\frac{a\,t}{L^2} - 12.0$, or $t - \frac{12.0 \times 22}{1.0}$ 48 minutes.

If the rate of change of temperature at any location within a slab or rod being heated or cooled is determined, the value of H can be determined by reversing the procedure just described.

Appendix IA
Heat Flow Tables for Infinite Plate or Slab (After Russell)

The designation N.E. indicates that in the calculation the standard procedure of using eight terms in the summation was not enough to give a value precise to five significant figures.

Plate. $HL = 0.2$

τ	Value of U when x/L is
	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99983	0.99932	0.99770	0.99349	0.98334	0.96885
0.04	N.E.	0.99829	0.99606	0.99173	0.98447	0.97240	0.95641
0.05	0.99995	0.99699	0.99396	0.98871	0.98040	0.96782	0.95147
0.06	0.99984	0.99551	0.99171	0.98567	0.97660	0.96362	0.94703
0.08	0.99938	0.99216	0.98713	0.97981	0.96965	0.95610	0.93923
0.10	0.99846	0.98859	0.98257	0.97431	0.96338	0.94944	0.93244
0.20	0.98826	0.97015	0.96155	0.95097	0.93831	0.92349	0.90646
0.30	0.97298	0.95204	0.94264	0.93142	0.91834	0.90340	0.88659
0.40	0.95601	0.93434	0.92478	0.91345	0.90039	0.88557	0.86905
0.50	0.93865	0.91699	0.90748	0.89626	0.88334	0.86876	0.85252
0.60	0.92138	0.89996	0.89059	0.87953	0.86683	0.85250	0.83656
0.70	0.90432	0.88326	0.87404	0.86318	0.85069	0.83663	0.82098
0.80	0.88756	0.86686	0.85780	0.84715	0.83490	0.82108	0.80572
0.90	0.87109	0.85077	0.84188	0.83141	0.81939	0.80583	0.79076
1.0	0.85493	0.83498	0.82626	0.81598	0.80418	0.79087	0.77608
1.5	0.77848	0.76032	0.75237	0.74302	0.73227	0.72015	0.70668
2.0	0.70886	0.69233	0.68509	0.67657	0.66679	0.65575	0.64349
3.0	0.58776	0.57404	0.56805	0.56098	0.55287	0.54372	0.53355
4.0	0.48734	0.47597	0.47100	0.46514	0.45841	0.45083	0.44240
5.0	0.40408	0.39465	0.39053	0.38567	0.38009	0.37380	0.36681
6.0	0.33504	0.32723	0.32381	0.31978	0.31516	0.30994	0.30414
7.0	0.27780	0.27132	0.26848	0.26515	0.26131	0.25698	0.25218
8.0	0.23034	0.22445	0.22261	0.21985	0.21667	0.21308	0.20910
9.0	0.19099	0.18653	0.18458	0.18229	0.17965	0.17668	0.17337
10.0	0.15836	0.15466	0.15305	0.15114	0.14930	0.14649	0.14375
11.0	0.13130	0.12824	0.12690	0.12532	0.12351	0.12146	0.11919
12.0	0.10887	0.10633	0.10522	0.10391	0.10241	0.10071	0.09883
16.0	0.05146	....	....	....	....	....	0.04671
20.0	0.02432	....	....	....	....	....	0.02208
24.0	0.01150	....	....	....	....	....	0.01043
28.0	0.00544	....	....	....	....	....	0.00493
32.0	0.00257	....	....	....	....	....	0.00233

Plate. $HL = 0.5$

τ	Value of U when x/L is
	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99960	0.99826	0.99424	0.98400	0.96200	0.92503
0.04	0.99992	0.99580	0.99044	0.98023	0.96267	0.93493	0.89645
0.05	0.99983	0.99273	0.98542	0.97292	0.95326	0.92404	0.88537
0.06	0.99959	0.98919	0.98014	0.96578	0.94394	0.91427	0.87551
0.08	0.99846	0.98131	0.96942	0.95222	0.92852	0.89718	0.85845
0.10	0.99630	0.97298	0.95892	0.93970	0.91448	0.88242	0.84387
0.15	0.98663	0.95190	0.93434	0.91210	0.88478	0.85203	0.81411
0.20	0.97257	0.93134	0.91196	0.88829	0.86012	0.82734	0.79013
0.30	0.93812	0.89197	0.87143	0.84703	0.81878	0.78671	0.75099
0.40	0.90103	0.85456	0.83420	0.81024	0.78274	0.75180	0.71754
0.50	0.86409	0.81880	0.79907	0.77591	0.74942	0.71970	0.68686
0.60	0.82823	0.78457	0.76555	0.74334	0.71790	0.68940	0.65793
0.70	0.79371	0.75179	0.73356	0.71773	0.68784	0.66051	0.63036
0.80	0.76058	0.72038	0.70291	0.68245	0.65907	0.63289	0.60400
0.90	0.72881	0.69028	0.67354	0.65393	0.63153	0.60644	0.57876
1.00	0.69837	0.66144	0.64540	0.62661	0.60514	0.58110	0.55457
2.00	0.45577	0.43167	0.42120	0.40894	0.39493	0.37923	0.36192
3.00	0.29744	0.28171	0.27488	0.26688	0.25774	0.24750	0.23620
4.00	0.19412	0.18385	0.17939	0.17417	0.16820	0.16152	0.15415
5.00	0.12668	0.11998	0.11707	0.11367	0.10977	0.10541	0.10060
6.00	0.08268	0.07830	0.07641	0.07418	0.07164	0.06879	0.06565
7.00	0.05396	0.05110	0.04986	0.04841	0.04675	0.04490	0.04285
8.00	0.03521	0.03335	0.03254	0.03159	0.03051	0.02930	0.02796
9.00	0.02298	0.02177	0.02124	0.02062	0.01991	0.01912	0.01825
10.00	0.01500	0.01420	0.01386	0.01346	0.01300	0.01248	0.01191
12.00	0.00639	....	....	....	....	....	0.00507
14.00	0.00272	....	....	....	....	....	0.00216
15.00	0.00178	....	....	....	....	....	0.00141

Plate. $HL = 1$

			Value of U when x/L is			
τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.05	0.99994	0.98654	0.97338	0.94963	0.91333	0.86080	0.79062
0.10	0.99332	0.95073	0.92539	0.89106	0.84643	0.79074	0.72384
0.20	0.94864	0.87950	0.84637	0.80627	0.75904	0.70481	0.64365
0.30	0.89202	0.81550	0.78183	0.74215	0.69663	0.64552	0.58912
0.40	0.83119	0.75696	0.72478	0.68718	0.64439	0.59670	0.54444
0.50	0.77455	0.70285	0.67269	0.63755	0.59765	0.55328	0.50479
0.60	0.71793	0.65269	0.62459	0.59188	0.55478	0.51356	0.46853
0.70	0.66681	0.60612	0.57999	0.54961	0.51513	0.47685	0.43500
0.80	0.61929	0.56290	0.53863	0.51039	0.47837	0.44282	0.40398
1.0	0.53411	0.48547	0.46454	0.44018	0.41257	0.38190	0.34841
1.5	0.36895	0.33535	0.32090	0.30407	0.28499	0.26380	0.24068
2.0	0.25487	0.23166	0.22167	0.21005	0.19687	0.18223	0.16625
3.0	0.12162	0.11054	0.10578	0.10023	0.09394	0.08696	0.07933
4.0	0.05803	0.05275	0.05047	0.04783	0.04483	0.04150	0.03786
5.0	0.02769	0.02517	0.02408	0.02282	0.02139	0.01980	0.01806
6.0	0.01321	....	....	....	....	....	0.00862
7.0	0.00630	....	....	....	....	....	0.00411
8.0	0.00301	....	....	....	....	....	0.00196
9.0	0.00144	....	....	....	....	....	0.00094

Plate. $HL = 2$

			Value of U when x/L is			
τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99999	0.99860	0.99403	0.97973	0.94373	0.87000	0.74655
0.04	0.99990	0.98552	0.96732	0.93327	0.87606	0.78943	0.66956
0.05	0.99954	0.97533	0.95117	0.91058	0.84811	0.75954	0.64380
0.06	0.99867	0.96395	0.93476	0.88925	0.82324	0.73390	0.62091
0.08	0.99485	0.93979	0.90294	0.85073	0.78067	0.69135	0.58350
0.10	0.98779	0.91543	0.87334	0.81710	0.74524	0.65717	0.55361
0.15	0.95813	0.85809	0.80921	0.74884	0.67655	0.59262	0.49795
0.20	0.91789	0.80643	0.75574	0.69517	0.62488	0.54541	0.45765
0.30	0.83661	0.71549	0.66699	0.61041	0.54632	0.47537	0.39841
0.40	0.74225	0.63652	0.59358	0.54128	0.48381	0.42059	0.35235
0.50	0.66025	0.56664	0.52709	0.48141	0.43012	0.37382	0.31314
0.60	0.58775	0.50455	0.46927	0.42854	0.38284	0.33269	0.27868
0.70	0.52335	0.44929	0.41785	0.38157	0.34087	0.29622	0.24812
0.80	0.46603	0.40009	0.37209	0.33978	0.30353	0.26376	0.22094
0.90	0.41500	0.35629	0.33135	0.30257	0.27029	0.23488	0.19675
1.00	0.36956	0.31654	0.29507	0.26944	0.24070	0.20916	0.17522
1.20	0.29306	0.25159	0.23399	0.21367	0.19087	0.16588	0.13899
1.40	0.23187	0.19952	0.18555	0.16905	0.15136	0.13154	0.11018
1.50	0.20695	0.17767	0.16524	0.15089	0.13479	0.11713	0.09812
1.60	0.18429	0.15822	0.14714	0.13437	0.12003	0.10431	0.08737
1.80	0.14615	0.12547	0.11669	0.10655	0.09519	0.08272	0.06929
2.00	0.11589	0.09950	0.09253	0.08450	0.07548	0.06559	0.05495
3.00	0.03634	....	....	....	....	....	0.01723
4.00	0.01140	....	....	....	....	....	0.00540
5.00	0.00357	....	....	....	....	....	0.00169
6.00	0.00112	....	....	....	....	....	0.00053

Plate. $HL = 3$
—Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	N.E.	0.99159	0.98748	0.92183	0.82292	0.65643
0.04	0.99981	0.98165	0.95531	0.92181	0.83343	0.72154	0.56786
0.05	0.99929	0.96739	0.93398	0.88107	0.79830	0.68554	0.53766
0.06	0.99810	0.95207	0.91273	0.85347	0.76771	0.65533	0.51263
0.08	0.99289	0.92049	0.87239	0.80562	0.71666	0.60665	0.47287
0.10	0.98350	0.88967	0.83586	0.76492	0.67536	0.56850	0.44208
0.15	0.94503	0.81967	0.75937	0.68558	0.59828	0.50126	0.38702
0.20	0.89480	0.75872	0.69777	0.62552	0.54258	0.45102	0.34885
0.30	0.78624	0.65488	0.59869	0.53368	0.45765	0.38161	0.29473
0.40	0.68153	0.56731	0.51779	0.46085	0.39726	0.32887	0.25384
0.50	0.59422	0.49193	0.44879	0.39927	0.34405	0.28466	0.21953
0.60	0.51559	0.42667	0.38922	0.34623	0.29832	0.24681	0.19051
0.70	0.44728	0.37011	0.33762	0.30031	0.25875	0.21369	0.16524
0.80	0.38800	0.32105	0.29286	0.26050	0.22444	0.18611	0.14331
0.90	0.33657	0.27850	0.25404	0.22597	0.19469	0.16107	0.12433
1.0	0.29196	0.24158	0.22037	0.19647	0.16888	0.13972	0.10785
1.2	0.21969	0.18179	0.16582	0.14750	0.12708	0.10513	0.08115
1.4	0.16531	0.13679	0.12478	0.11099	0.09563	0.07911	0.06107
1.5	0.14340	0.11866	0.10799	0.09628	0.08295	0.06863	0.05297
1.6	0.12411	0.10293	0.09389	0.08352	0.07196	0.05953	0.04595
1.8	0.09360	0.07745	0.07065	0.06284	0.05415	0.04480	0.03458
2.0	0.07044	0.05825	0.05316	0.04729	0.04074	0.03371	0.02602
3.0	0.01699						0.00628
4.0	0.00410						0.00152
5.0	0.00099						0.00037

Plate. $HL = 4$
—Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99554	0.98860	0.96445	0.90289	0.78040	0.58354
0.04	0.99975	0.97393	0.93491	0.88837	0.79900	0.66522	0.48925
0.05	0.99917	0.95774	0.91940	0.85563	0.75906	0.62586	0.45839
0.06	0.99773	0.94016	0.89464	0.82461	0.72483	0.59339	0.43330
0.08	0.99138	0.90410	0.84844	0.77093	0.66887	0.54325	0.39440
0.10	0.98013	0.86924	0.80743	0.72632	0.62463	0.50406	0.36499
0.15	0.93529	0.78157	0.72370	0.64119	0.54431	0.43514	0.31383
0.20	0.87786	0.72534	0.65786	0.57843	0.48784	0.38818	0.27941
0.30	0.75740	0.61464	0.55406	0.48435	0.40646	0.32224	0.23160
0.40	0.64744	0.52306	0.47078	0.41094	0.34442	0.27280	0.19599
0.50	0.55218	0.44559	0.40092	0.34982	0.29310	0.23210	0.16672
0.60	0.47067	0.37970	0.34160	0.29804	0.24969	0.19772	0.14201
0.70	0.40113	0.32358	0.29110	0.25397	0.21276	0.16847	0.12102
0.80	0.34185	0.27576	0.24808	0.21643	0.18132	0.14358	0.10313
0.90	0.29133	0.23501	0.21142	0.18445	0.15452	0.12236	0.08789
1.00	0.24828	0.20028	0.18017	0.15719	0.13169	0.10427	0.07490
1.20	0.18032	0.14546	0.13085	0.11416	0.09564	0.07573	0.05451
1.40	0.13096	0.10564	0.09503	0.08291	0.06946	0.05500	0.03967
1.50	0.11160	0.09003	0.08099	0.07066	0.05919	0.04687	0.03381
2.0	0.05012	0.04047	0.03641	0.03176	0.02660	0.02107	0.01520
2.5	0.02255	0.01819	0.01637	0.01428	0.01196	0.00947	0.00681
3.0	0.01014						0.00306
4.0	0.00205						0.00062

Plate. $HL = 4.8$
—Value of U when x/L is—

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	N.E.	0.99540	0.96625	0.88998	0.74562	0.52177
0.04	N.E.	0.97244	0.93971	0.87785	0.77588	0.62819	0.43506
0.05	0.99969	0.95406	0.91074	0.83992	0.73334	0.58873	0.40594
0.06	0.99767	0.93439	0.88357	0.80620	0.69724	0.55430	0.38219
0.08	0.99039	0.89481	0.83352	0.74899	0.63898	0.50370	0.34540
0.10	0.97784	0.85715	0.78984	0.70221	0.59352	0.46463	0.31790
0.15	0.92890	0.77538	0.70213	0.61448	0.51346	0.39708	0.27086
0.20	0.86723	0.70545	0.63423	0.55077	0.45617	0.35182	0.23963
0.30	0.74008	0.59126	0.52840	0.45626	0.37602	0.28894	0.19661
0.40	0.62598	0.49791	0.44433	0.38313	0.31539	0.24213	0.16472
0.50	0.52818	0.41977	0.37447	0.32279	0.26564	0.20390	0.13870
0.60	0.44551	0.35398	0.31576	0.27216	0.22395	0.17189	0.11692
0.70	0.37574	0.29852	0.26628	0.22951	0.18885	0.14495	0.09860
0.80	0.31688	0.25175	0.22457	0.19356	0.15927	0.12224	0.08315
0.90	0.26724	0.21281	0.18939	0.16324	0.13431	0.10309	0.07013
1.0	0.22537	0.17906	0.15972	0.13766	0.11327	0.08694	0.05914
1.2	0.16030	0.12735	0.11360	0.09791	0.08057	0.06184	0.04206
1.4	0.11401	0.09058	0.08080	0.06980	0.05730	0.04398	0.02992
1.5	0.09615	0.07639	0.06814	0.05873	0.04832	0.03709	0.02522
1.6	0.08109	0.06442	0.05747	0.04953	0.04075	0.03128	0.02128
1.8	0.05767	0.04582	0.04087	0.03523	0.02899	0.02225	0.01513
2.0	0.04102	0.03259	0.02907	0.02506	0.02062	0.01582	0.01076
2.5	0.01750	....	....	....	....	....	0.00459
3.0	0.00746	....	....	....	....	....	0.00196
3.5	0.00318	....	....	....	....	....	0.00084
4.0	0.00136	....	....	....	....	....	0.00036

Plate. $HL = 6$
—Value of U when x/L is—

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99989	0.99638	0.98579	0.95262	0.87257	0.71286	0.47260
0.04	0.99976	0.96973	0.92937	0.85976	0.74707	0.58697	0.37845
0.05	0.99891	0.94905	0.89849	0.81923	0.70199	0.54518	0.34946
0.06	0.99602	0.92723	0.86884	0.78316	0.66403	0.51137	0.32654
0.08	0.98893	0.88346	0.81532	0.72268	0.60364	0.45944	0.29200
0.10	0.97496	0.84264	0.76873	0.67390	0.55732	0.42081	0.26675
0.15	0.92114	0.74498	0.68716	0.58402	0.47619	0.35557	0.22455
0.20	0.85467	0.68276	0.60748	0.51983	0.42101	0.31280	0.19713
0.30	0.72031	0.56530	0.50012	0.42560	0.34305	0.25391	0.15981
0.40	0.60184	0.47047	0.41538	0.35332	0.28447	0.21038	0.13237
0.50	0.50190	0.39201	0.34621	0.29422	0.23684	0.17512	0.11017
0.60	0.41840	0.32670	0.28857	0.24518	0.19735	0.14591	0.09179
0.70	0.34873	0.27231	0.24052	0.20435	0.16448	0.12161	0.07651
0.80	0.29067	0.22469	0.20047	0.17032	0.13713	0.10136	0.06377
0.90	0.24227	0.18917	0.16709	0.14197	0.11426	0.08448	0.05310
1.00	0.20193	0.15768	0.13927	0.11833	0.09511	0.07042	0.04430
1.20	0.14028	0.10954	0.09675	0.08220	0.06617	0.04892	0.03078
1.40	0.09746	0.07610	0.06721	0.05711	0.04597	0.03398	0.02138
1.50	0.08123	0.06343	0.05602	0.04760	0.03831	0.02833	0.01782
2.0	0.03268	0.02551	0.02360	0.01915	0.01541	0.01139	0.00717
2.5	0.01314	0.01026	0.00949	0.00788	0.00620	0.00458	0.00288
3.0	0.00529						0.00116
3.5	0.00213						0.00047

Plate. $HL = 7.5$

Value of U when x/L is—

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99590	0.98327	0.94527	0.85554	0.67962	0.41207
0.04	0.99974	0.96350	0.92024	0.84283	0.71994	0.54403	0.32161
0.05	0.99875	0.94082	0.88648	0.79925	0.67251	0.50161	0.29483
0.06	0.99664	0.91710	0.85443	0.76095	0.63294	0.46798	0.27398
0.08	0.98753	0.87045	0.79719	0.69780	0.57110	0.41736	0.24303
0.10	0.97205	0.82738	0.74858	0.64754	0.52444	0.38038	0.22077
0.15	0.91369	0.73623	0.65436	0.55657	0.44414	0.31884	0.18420
0.20	0.84293	0.66205	0.58358	0.49251	0.39026	0.27890	0.16078
0.30	0.70245	0.54242	0.47547	0.39915	0.31488	0.22424	0.12908
0.40	0.58062	0.44676	0.39117	0.32802	0.25851	0.18397	0.10586
0.50	0.47911	0.36839	0.32246	0.27034	0.21300	0.15157	0.08722
0.60	0.39522	0.30383	0.26595	0.22294	0.17525	0.12499	0.07191
0.70	0.32593	0.25117	0.21935	0.18388	0.14488	0.10309	0.05931
0.80	0.26887	0.20669	0.18092	0.15167	0.11949	0.08503	0.04892
0.90	0.22177	0.17009	0.14922	0.12510	0.09856	0.07013	0.04035
1.0	0.18292	0.14062	0.12308	0.10318	0.08129	0.05784	0.03328
1.2	0.12431	0.09566	0.08373	0.07019	0.05530	0.03935	0.02264
1.4	0.08466	0.06508	0.05696	0.04775	0.03762	0.02677	0.01540
1.5	0.06983	0.05368	0.04698	0.03943	0.03103	0.02208	0.01270
1.6	0.05759	0.04427	0.03875	0.03256	0.02560	0.01821	0.01048
1.8	0.03918	0.03012	0.02636	0.02210	0.01741	0.01239	0.00713
2.0	0.02665	0.02049	0.01794	0.01503	0.01185	0.00843	0.00485
2.5	0.01017	....	....	....	....	....	0.00185
3.0	0.00389	....	....	....	....	....	0.00071
3.5	0.00148	....	....	....	....	....	0.00027

Plate. $HL = 8$

Value of U when x/L is—

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99581	0.98416	0.94308	0.84881	0.66905	0.39421
0.04	0.99973	0.96226	0.91872	0.83798	0.71082	0.53225	0.30596
0.05	0.99871	0.93895	0.88398	0.79360	0.66257	0.48987	0.27997
0.06	0.99648	0.91464	0.85117	0.75485	0.62289	0.45688	0.25979
0.08	0.98714	0.86709	0.79280	0.69092	0.56107	0.40619	0.22998
0.10	0.97101	0.82337	0.74348	0.64037	0.51464	0.36973	0.20862
0.15	0.91163	0.73125	0.64840	0.54925	0.43502	0.30924	0.17368
0.20	0.83971	0.65656	0.57732	0.48531	0.38182	0.27012	0.15138
0.30	0.69769	0.53642	0.46905	0.39228	0.30724	0.21664	0.12124
0.40	0.57503	0.44162	0.38484	0.32150	0.25157	0.17727	0.09917
0.50	0.47318	0.36231	0.31637	0.26425	0.20673	0.14566	0.08148
0.60	0.38923	0.29797	0.26019	0.21731	0.17000	0.11977	0.06700
0.70	0.32015	0.24509	0.21401	0.17874	0.13983	0.09851	0.05511
0.80	0.26333	0.20160	0.17603	0.14702	0.11501	0.08103	0.04533
0.90	0.21660	0.16582	0.14445	0.12093	0.09460	0.06664	0.03728
1.0	0.17825	0.13639	0.11909	0.09946	0.07781	0.05482	0.03067
1.5	0.06707	0.05135	0.04483	0.03744	0.02929	0.02064	0.01154
2.0	0.02525	0.01933	0.01684	0.01409	0.01102	0.00776	0.00435
2.5	0.00951	....	....	....	....	....	0.00164
3.0	0.00358	....	....	....	....	....	0.00062
3.5	0.00135	....	....	....	....	....	0.00023

Plate. $HL = 10$
—Value of U when x/L is—

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99502	0.97971	0.93536	0.83007	0.63234	0.33641
0.04	0.99970	0.95774	0.90844	0.82257	0.68464	0.49340	0.25538
0.05	0.99856	0.93241	0.87118	0.77531	0.63497	0.45157	0.23230
0.06	0.99609	0.90630	0.83664	0.73464	0.59462	0.41895	0.21462
0.08	0.98573	0.85715	0.77587	0.66864	0.53259	0.37063	0.18880
0.10	0.96842	0.81016	0.72516	0.61727	0.48656	0.33597	0.17057
0.15	0.90476	0.71520	0.62863	0.52602	0.40869	0.27924	0.14110
0.20	0.82925	0.63905	0.55716	0.46268	0.35717	0.24289	0.12248
0.30	0.68246	0.51755	0.44888	0.37093	0.28514	0.19331	0.09735
0.40	0.55742	0.42142	0.36515	0.30145	0.23254	0.15724	0.07897
0.50	0.45464	0.34351	0.29557	0.24563	0.18863	0.12789	0.06432
0.60	0.37071	0.28005	0.24259	0.20023	0.15376	0.10416	0.05242
0.70	0.30225	0.22834	0.19779	0.16325	0.12537	0.08493	0.04275
0.80	0.24643	0.18617	0.16127	0.13310	0.10221	0.06924	0.03486
0.90	0.20092	0.15179	0.13148	0.10852	0.08333	0.05646	0.02842
1.0	0.16382	0.12404	0.10720	0.08847	0.06795	0.04603	0.02317
1.2	0.10890	0.08227	0.07126	0.05882	0.04517	0.03053	0.01540
1.4	0.07239	0.05469	0.04737	0.03910	0.03003	0.02029	0.01024
1.5	0.05902	0.04459	0.03862	0.03188	0.02448	0.01659	0.00835
1.6	0.04812	0.03635	0.03149	0.02599	0.01996	0.01353	0.00680
1.8	0.03199	0.02417	0.02094	0.01728	0.01326	0.00899	0.00453
2.0	0.02126	0.01606	0.01392	0.01149	0.00882	0.00598	0.00301
2.5	0.00766	....	....	....	....	....	0.00108
3.0	0.00276	....	....	....	....	....	0.00039
3.5	0.00099	....	....	....	....	....	0.00014

Plate. $HL = 13$
—Value of U when x/L is-

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99430	0.97708	0.92684	0.80968	0.58904	0.27382
0.04	0.99985	0.95272	0.89836	0.80536	0.65688	0.45219	0.20362
0.05	0.99855	0.92522	0.85870	0.75424	0.60625	0.41177	0.18421
0.06	0.99570	0.89722	0.82209	0.71242	0.56511	0.38057	0.16949
0.08	0.98429	0.84402	0.76883	0.64530	0.50393	0.33488	0.14831
0.10	0.96549	0.79642	0.70678	0.59364	0.45876	0.30249	0.13350
0.15	0.89761	0.69897	0.60908	0.50295	0.38321	0.25008	0.10984
0.20	0.81855	0.62165	0.53743	0.44054	0.33361	0.21673	0.09501
0.30	0.66730	0.49913	0.42942	0.35039	0.26433	0.17124	0.07497
0.40	0.54018	0.40295	0.34636	0.28238	0.21287	0.13783	0.06033
0.50	0.43672	0.32561	0.27983	0.22812	0.17194	0.11132	0.04872
0.60	0.35300	0.26317	0.22616	0.18436	0.13895	0.08996	0.03937
0.70	0.28532	0.21271	0.18280	0.14901	0.11231	0.07271	0.03182
0.80	0.23061	0.17192	0.14774	0.12016	0.09078	0.05877	0.02572
0.90	0.18639	0.13896	0.11942	0.09734	0.07337	0.04750	0.02079
1.00	0.15065	0.11231	0.09652	0.07868	0.05930	0.03839	0.01680
1.20	0.09840	0.07337	0.06305	0.05140	0.03874	0.02508	0.01097
1.40	0.06430	0.04793	0.04119	0.03358	0.02531	0.01638	0.00717
1.50	0.05197	0.03874	0.03329	0.02714	0.02046	0.01324	0.00580
1.60	0.04200	0.03131	0.02691	0.02194	0.01653	0.01070	0.00468
1.80	0.02744	0.02546	0.01758	0.01433	0.01080	0.00699	0.00306
2.00	0.01793	0.01336	0.01148	0.00936	0.00705	0.00457	0.00200
2.50	0.00618						0.00069
3.00	0.00213						0.00024

Plate. $HL = 14$

Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99996	0.99402	0.97613	0.92370	0.80263	0.58054	0.25691
0.04	0.99959	0.95128	0.89561	0.79844	0.64823	0.44197	0.19034
0.05	0.99826	0.92322	0.85530	0.74884	0.59748	0.40185	0.17205
0.06	0.99535	0.89476	0.81825	0.70674	0.55690	0.37102	0.15820
0.08	0.98371	0.84085	0.75440	0.63936	0.49547	0.32604	0.13827
0.10	0.96452	0.79282	0.70206	0.58767	0.45061	0.29426	0.12464
0.15	0.89563	0.69480	0.60413	0.49721	0.37586	0.24295	0.10224
0.20	0.81572	0.61725	0.53249	0.43506	0.32687	0.21035	0.08836
0.30	0.66582	0.49454	0.42459	0.34534	0.25848	0.16589	0.06960
0.40	0.53619	0.39838	0.34174	0.27773	0.20774	0.13326	0.05589
0.50	0.43233	0.32123	0.27551	0.22387	0.16743	0.10729	0.04504
0.60	0.34866	0.25906	0.22219	0.18054	0.13502	0.08660	0.03632
0.70	0.28120	0.20894	0.17919	0.14561	0.10889	0.06984	0.02929
0.80	0.22679	0.16851	0.14452	0.11743	0.08782	0.05633	0.02363
0.90	0.18291	0.13590	0.11656	0.09471	0.07083	0.04543	0.01905
1.00	0.14751	0.10961	0.09400	0.07639	0.05713	0.03664	0.01537
1.20	0.09595	0.07129	0.06115	0.04968	0.03716	0.02383	0.00999
1.40	0.06241	0.04637	0.03977	0.03232	0.02417	0.01550	0.00650
1.50	0.05034	0.03740	0.03208	0.02606	0.01949	0.01250	0.00524
1.60	0.04060	0.03016	0.02587	0.02102	0.01572	0.01008	0.00423
1.80	0.02641	0.01962	0.01683	0.01367	0.01023	0.00656	0.00275
2.0	0.01718	0.01276	0.01095	0.00889	0.00665	0.00427	0.00179
2.5	0.00586	....	....	....	....	....	0.00061
3.0	0.00200	....	....	....	....	....	0.00021

Plate. $HL = 17$

Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99997	0.99339	0.97385	0.91726	0.78791	0.55350	0.21814
0.04	0.99955	0.94777	0.88873	0.78653	0.62982	0.41667	0.15953
0.05	0.99811	0.91836	0.84694	0.73578	0.57889	0.37779	0.14375
0.06	0.99500	0.88880	0.80885	0.69327	0.53845	0.34808	0.13188
0.08	0.98262	0.83330	0.74371	0.62552	0.47768	0.30505	0.11494
0.10	0.96248	0.78427	0.69077	0.57361	0.43362	0.27483	0.10322
0.15	0.89108	0.68503	0.59249	0.48369	0.36062	0.22630	0.08462
0.20	0.80913	0.60696	0.51657	0.42226	0.31296	0.19562	0.07300
0.30	0.65441	0.48387	0.41279	0.33367	0.24642	0.15359	0.05726
0.40	0.52580	0.38783	0.33099	0.26703	0.19707	0.12278	0.04577
0.50	0.42202	0.31113	0.26556	0.21416	0.15805	0.09845	0.03670
0.60	0.33863	0.24964	0.21307	0.17183	0.12680	0.07899	0.02944
0.70	0.27172	0.20031	0.17097	0.13787	0.10175	0.06338	0.02363
0.80	0.21803	0.16073	0.13719	0.11063	0.08164	0.05086	0.01896
0.90	0.17495	0.12897	0.11008	0.08877	0.06551	0.04081	0.01521
1.0	0.14038	0.10349	0.08833	0.07123	0.05256	0.03274	0.01221
1.2	0.09038	0.06663	0.05687	0.04586	0.03384	0.02108	0.00786
1.4	0.05819	0.04290	0.03662	0.02953	0.02179	0.01357	0.00506
1.5	0.04669	0.03442	0.02938	0.02369	0.01748	0.01089	0.00406
1.6	0.03747	0.02762	0.02357	0.01901	0.01403	0.00874	0.00326
1.8	0.02412	0.01778	0.01518	0.01224	0.00903	0.00563	0.00210
2.0	0.01553	0.01145	0.00977	0.00788	0.00582	0.00362	0.00135
2.2	0.01000	....	....	....	....	....	0.00087
2.4	0.00644	....	....	....	....	....	0.00056
2.6	0.00415	....	....	....	....	....	0.00036
2.8	0.00267	....	....	....	....	....	0.00023
3.0	0.00172	....	....	....	....	....	0.00015

Plate. HL = 20

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99305	0.97178	0.91204	0.77812	0.53165	N.E.
0.04	0.99986	0.94516	0.88336	0.77712	0.61562	0.39720	0.13618
0.05	0.99831	0.91470	0.84046	0.72563	0.56458	0.35941	0.12272
0.06	0.99498	0.88428	0.80160	0.68250	0.52436	0.33071	0.11234
0.08	0.98203	0.82760	0.73564	0.62445	0.46432	0.28931	0.09786
0.10	0.96112	0.77789	0.68235	0.56493	0.42000	0.26038	0.08787
0.15	0.88778	0.67787	0.58396	0.47383	0.34957	0.21409	0.07204
0.20	0.80434	0.59948	0.51226	0.41303	0.30298	0.18480	0.06212
0.30	0.64788	0.47621	0.40538	0.32533	0.23766	0.14472	0.04861
0.40	0.51858	0.38030	0.32350	0.25944	0.18958	0.11530	0.03873
0.50	0.41466	0.30398	0.25854	0.20733	0.15148	0.09212	0.03094
0.60	0.33152	0.24300	0.20668	0.16573	0.12109	0.07364	0.02473
0.70	0.26503	0.19427	0.16523	0.13249	0.09680	0.05887	0.01977
0.80	0.21187	0.15531	0.13209	0.10592	0.07739	0.04706	0.01581
0.90	0.16938	0.12390	0.10560	0.08468	0.06187	0.03762	0.01264
1.00	0.13541	0.09926	0.08442	0.06769	0.04946	0.03008	0.01010
1.20	0.08654	0.06343	0.05395	0.04327	0.03161	0.01922	0.00645
1.40	0.05531	0.04054	0.03448	0.02765	0.02020	0.01228	0.00413
1.60	0.03535	0.02591	0.02204	0.01769	0.01291	0.00785	0.00264
1.80	0.02259	0.01656	0.01408	0.01129	0.00825	0.00502	0.00168
2.0	0.01444	0.01058	0.00900	0.00722	0.00528	0.00321	0.00108
2.2	0.00930	....	....	....	....	....	0.00069
2.4	0.00590	....	....	....	....	....	0.00044
2.6	0.00377	....	....	....	....	....	0.00028
2.8	0.00241	....	....	....	....	....	0.00018

Plate. HL = 32

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99998	0.99152	0.96716	0.89865	0.74697	0.48303	0.12180
0.04	0.99943	0.93824	0.87049	0.75568	0.58353	0.35516	0.08711
0.05	0.99767	0.90556	0.82536	0.70283	0.53316	0.32025	0.07809
0.06	0.99397	0.87341	0.78505	0.65909	0.49380	0.29394	0.07140
0.08	0.97972	0.81444	0.71753	0.59094	0.43557	0.25633	0.06196
0.10	0.95720	0.76338	0.66367	0.53989	0.39393	0.23022	0.05548
0.15	0.87976	0.66192	0.56518	0.45241	0.32578	0.18866	0.04530
0.20	0.79322	0.58295	0.49424	0.39299	0.28147	0.16241	0.03893
0.30	0.63327	0.45941	0.38793	0.30731	0.21943	0.12635	0.03026
0.40	0.50267	0.36394	0.30710	0.24314	0.17353	0.09989	0.02392
0.50	0.39865	0.28853	0.24345	0.19272	0.13753	0.07917	0.01896
0.60	0.31611	0.22878	0.19303	0.15281	0.10905	0.06277	0.01503
0.70	0.25065	0.18140	0.15306	0.12117	0.08677	0.04977	0.01192
0.80	0.19875	0.14384	0.12136	0.09608	0.06856	0.03946	0.00945
0.90	0.15759	0.11405	0.09623	0.07618	0.05437	0.03129	0.00749
1.00	0.12496	0.09044	0.07631	0.06041	0.04311	0.02481	0.00594
1.20	0.07857	0.05686	0.04798	0.03798	0.02711	0.01560	0.00374
1.40	0.04940	0.03575	0.03016	0.02388	0.01704	0.00981	0.00235
1.50	0.03917	0.02835	0.02392	0.01893	0.01351	0.00778	0.00186
1.60	0.03106	0.02248	0.01897	0.01501	0.01071	0.00617	0.00148
1.80	0.01953	0.01413	0.01192	0.00944	0.00674	0.00388	0.00093
2.00	0.01228	0.00889	0.00750	0.00748	0.00423	0.00244	0.00058
2.20	0.00772	....	....	....	....	....	0.00037
2.40	0.00485	....	....	....	....	....	0.00023
2.60	0.00305	....	....	....	....	....	0.00015
2.80	0.00192	.. ....	. .....	....	....	....	0.00009

Value of U when x/L is

Plate. $HL = 44$

–Value of U when x/L is——

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99998	0.99069	0.96440	0.89130	0.73350	0.45777	0.08952
0.04	0.99938	0.93461	0.86377	0.74464	0.56749	0.33453	0.06372
0.05	0.99752	0.90084	0.81761	0.69130	0.51762	0.30125	0.05705
0.06	0.99358	0.86789	0.77668	0.64744	0.47882	0.27625	0.05212
0.08	0.97865	0.80789	0.70859	0.57945	0.42171	0.24060	0.04519
0.10	0.95528	0.75627	0.65456	0.52878	0.38102	0.21593	0.04044
0.15	0.87584	0.65422	0.55630	0.44228	0.31462	0.17675	0.03299
0.20	0.78784	0.57511	0.48559	0.38360	0.27147	0.15199	0.02833
0.30	0.62634	0.45154	0.37978	0.29896	0.21096	0.11788	0.02196
0.40	0.49520	0.35634	0.29953	0.23566	0.16621	0.09286	0.01728
0.50	0.39120	0.28142	0.23653	0.18607	0.13124	0.07331	0.01365
0.60	0.30900	0.22228	0.18682	0.14696	0.10365	0.05790	0.01078
0.70	0.24407	0.17557	0.14756	0.11608	0.08187	0.04573	0.00851
0.80	0.19278	0.13867	0.11655	0.09169	0.06468	0.03613	0.00673
0.90	0.15227	0.10953	0.09206	0.07242	0.05107	0.02853	0.00531
1.00	0.12027	0.08651	0.07271	0.05720	0.04034	0.02254	0.00420
1.20	0.07503	0.05397	0.04536	0.03569	0.02517	0.01406	0.00262
1.40	0.04681	0.03367	0.02830	0.02226	0.01570	0.00877	0.00163
1.50	0.03698	0.02660	0.02235	0.01759	0.01240	0.00693	0.00129
1.60	0.02920	0.02103	0.01766	0.01389	0.00980	0.00547	0.00102
1.80	0.01822	0.01311	0.01102	0.00867	0.00611	0.00341	0.00064
2.00	0.01137	0.00818	0.00687	0.00541	0.00381	0.00213	0.00040
2.20	0.00709	....	....	....	....	....	0.00025
2.40	0.00443	....	....	....	....	....	0.00015
2.60	0.00276	....	....	....	....	....	0.00010
2.80	0.00172	....	....	....	....	....	0.00006

Plate. $HL = 60$

——Value of U when x/L is——

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99973	0.99003	0.96214	0.86081	0.72314	0.43942	0.06601
0.04	0.99933	0.93180	0.85861	0.73577	0.55561	0.31948	0.04685
0.05	0.99732	0.89697	0.81177	0.67270	0.49614	0.28748	0.04193
0.06	0.99328	0.86371	0.77041	0.63881	0.46785	0.26339	0.03837
0.08	0.97775	0.80299	0.70196	0.57103	0.41160	0.22925	0.03317
0.10	0.95379	0.75100	0.64787	0.52067	0.37166	0.20564	0.02968
0.15	0.87320	0.64861	0.54981	0.43494	0.30656	0.16822	0.02420
0.20	0.78386	0.56941	0.47933	0.37683	0.26428	0.14452	0.02076
0.30	0.62127	0.44586	0.37392	0.29296	0.20490	0.11185	0.01607
0.40	0.48978	0.35086	0.29409	0.23030	0.16101	0.08784	0.01262
0.50	0.38582	0.27632	0.23159	0.18134	0.12677	0.06917	0.00993
0.60	0.30389	0.21763	0.18240	0.14282	0.09984	0.05448	0.00782
0.70	0.23936	0.17142	0.14367	0.11249	0.07864	0.04291	0.00616
0.80	0.18853	0.13501	0.11316	0.08860	0.06194	0.03380	0.00486
0.90	0.14849	0.10634	0.08913	0.06979	0.04878	0.02662	0.00382
1.0	0.11696	0.08376	0.07183	0.05497	0.03842	0.02096	0.00301
1.2	0.07256	0.05196	0.04355	0.03410	0.02384	0.01301	0.00186
1.4	0.04501	0.03224	0.02702	0.02115	0.01479	0.00807	0.00116
1.5	0.03546	0.02539	0.02128	0.01666	0.01165	0.00636	0.00091
1.6	0.02792	0.02000	0.01676	0.01312	0.00917	0.00501	0.00072
1.8	0.01732	0.01241	0.01040	0.00814	0.00569	0.00311	0.00044
2.0	0.01074	0.00770	0.00645	0.00505	0.00353	0.00193	0.00028
2.2	0.00667	....	....	....	....	....	0.00017
2.4	0.00414	....	....	....	....	....	0.00011
2.6	0.00257	....	....	....	....	....	0.00007

Plate. $HL = \infty$

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
			Value of U when x/L is				
0.02	0.99967	0.98750	0.95432	0.86763	0.60871	0.38318	0
0.04	0.99915	0.92286	0.84271	0.71116	0.49894	0.27686	0
0.05	0.99695	0.88614	0.79409	0.65722	0.46127	0.24818	0
0.06	0.99221	0.85108	0.75174	0.61354	0.42991	0.22706	0
0.08	0.97516	0.78853	0.68007	0.54673	0.38108	0.19741	0
0.10	0.94931	0.73565	0.62756	0.49751	0.34469	0.17692	0
0.15	0.86429	0.63252	0.53099	0.41424	0.28402	0.14447	0
0.20	0.77231	0.55318	0.46154	0.35783	0.24424	0.12387	0
0.30	0.60680	0.42984	0.35749	0.27627	0.18812	0.09526	0
0.40	0.47440	0.33559	0.27898	0.21550	0.14669	0.07410	0
0.50	0.37078	0.26218	0.21794	0.16833	0.11458	0.05800	0
0.60	0.28971	0.20486	0.17029	0.13152	0.08953	0.04532	0
0.70	0.22636	0.16006	0.13305	0.10277	0.06995	0.03541	0
0.80	0.17687	0.12506	0.10396	0.08030	0.05466	0.02767	0
0.90	0.13819	0.09772	0.08123	0.06274	0.04271	0.02162	0
1.0	0.10798	0.07635	0.06347	0.04902	0.03337	0.01689	0
1.2	0.06592	0.04661	0.03875	0.02993	0.02037	0.01031	0
1.4	0.04024	0.02846	0.02365	0.01827	0.02144	....	0
1.5	0.03144	0.02223	0.01844	0.01428	....	....	0
2.0	0.00925	0.00647	0.00538	....	....	....	0
2.5	0.00267	0.00189	0.00157	....	....	....	0
3.0	0.00078	....	....	....	....	....	0

Appendix IB
Tables For Infinite Cylinder or Rod (After Russell)

Cylinder. $HL = 0.05$

	0.0	0.5	0.6	Value of U when x/L is 0.7	0.8	0.9	1.0
0.08	0.99941	0.99680	0.99520	0.99301	0.98987	0.98640	0.98189
0.10	0.99866	0.99522	0.99333	0.99089	0.98753	0.98393	0.97940
0.20	0.99172	0.98624	0.98374	0.98076	0.97699	0.97317	0.96859
0.30	0.98264	0.97673	0.97411	0.97103	0.96721	0.96335	0.95880
0.40	0.97313	0.96717	0.96455	0.96147	0.95765	0.95383	0.94932
0.50	0.96360	0.95767	0.95507	0.95201	0.94823	0.94444	0.93997
0.60	0.95415	0.94826	0.94568	0.94265	0.93891	0.93516	0.93073
0.80	0.93548	0.92972	0.92719	0.92422	0.92055	0.91687	0.91253
1.00	0.91719	0.91153	0.90905	0.90614	0.90254	0.89893	0.89468
1.20	0.89925	0.89371	0.89127	0.88842	0.88489	0.88135	0.87718
1.60	0.86442	0.85909	0.85675	0.85400	0.85061	0.84721	0.84320
2.00	0.83093	0.82581	0.82357	0.82093	0.81767	0.81440	0.81054
2.8	0.76781	0.76308	0.76100	0.75856	0.75555	0.75253	0.74897
3.0	0.75280	0.74815	0.74612	0.74373	0.74077	0.73781	0.73432
4.0	0.68200	0.67780	0.67595	0.67379	0.67111	0.66843	0.66527
5.0	0.61787	0.61406	0.61239	0.61042	0.60800	0.60557	0.60270
5.6	0.58232	0.57873	0.57715	0.57530	0.57302	0.57073	0.56803
6.0	0.55976	0.55631	0.55480	0.55302	0.55097	0.54862	0.54603
7.0	0.50712	0.50400	0.50262	0.50101	0.49902	0.49703	0.49468
8.0	0.45943	0.45660	0.45536	0.45390	0.45210	0.45029	0.44816
10.0	0.37708	0.37476	0.37374	0.37254	0.37106	0.36958	0.36783
12.0	0.30950	0.30759	0.30675	0.30577	0.30456	0.30334	0.30190
14.0	0.25402	0.25246	0.25177	0.25096	0.24997	0.24897	0.24779
16.0	0.20849	0.20721	0.20664	0.20598	0.20516	0.20434	0.20338
18.0	0.17112	0.17007	0.16961	0.16906	0.16839	0.16772	0.16692
20.0	0.14045	0.13959	0.13921	0.13876	0.13821	0.13766	0.13701
25	0.08572	....	....	....	....	....	0.08352
30	0.05231	....	....	....	....	....	0.05103
40	0.01949	....	....	....	....	....	0.01901
50	0.00726	....	....	....	....	....	0.00708
60	0.00270	....	....	....	....	....	0.00264

Cylinder. $HL = 0.1$

τ	0.0	0.5	0.6	Value of U when x/L is 0.7	0.8	0.9	1.0
0.1	0.99722	0.98946	0.98671	0.98313	0.97515	0.96920	0.95930
0.2	0.98356	0.97191	0.96796	0.96336	0.95467	0.94832	0.93847
0.3	0.96584	0.95344	0.94932	0.94455	0.93585	0.92951	0.91982
0.4	0.94747	0.93513	0.93099	0.92625	0.91768	0.91144	0.90192
0.5	0.92923	0.91709	0.91300	0.90834	0.89993	0.89380	0.88447
0.6	0.91129	0.89936	0.89536	0.89079	0.88254	0.87653	0.86738
0.8	0.87641	0.86494	0.86109	0.85670	0.84876	0.84298	0.83418
1.0	0.84287	0.83184	0.82814	0.82391	0.81628	0.81072	0.80225
1.5	0.76453	0.75452	0.75116	0.74733	0.74040	0.73520	0.72769
2.0	0.69346	0.68439	0.68134	0.67786	0.67159	0.66686	0.66005
3.0	0.57054	0.56307	0.56057	0.55770	0.55254	0.54865	0.54304
4.0	0.46940	0.46326	0.46120	0.45884	0.45459	0.45140	0.44678
5.0	0.38620	0.38114	0.37945	0.37751	0.37401	0.37138	0.36759
6.0	0.31774	0.31430	0.31219	0.31059	0.30771	0.30555	0.30243
8.0	0.21508	0.21226	0.21132	0.21024	0.20829	0.20678	0.20471
10.0	0.14559	0.14335	0.14304	0.14231	0.14099	0.13997	0.13857
12.0	0.09855	0.09726	0.09683	0.09633	0.09544	0.09496	0.09380
14.0	0.06671	0.06583	0.06554	0.06521	0.06460	0.06428	0.06349
16.0	0.04515	0.04456	0.04437	0.04414	0.04373	0.04351	0.04298
18.0	0.03056	0.03016	0.03003	0.02988	0.02960	0.02945	0.02909
20.0	0.02069	0.02042	0.02033	0.02022	0.02004	0.01994	0.01969
25.0	0.00780	....	....	....	....	....	0.00742
30.0	0.00294	....	....	....	....	....	0.00280
35.0	0.00111	....	....	....	....	....	0.00106

Cylinder. $HL = 0.2$

				Value of U when x/L is			
τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.99381	0.98051	0.97330	0.96374	0.95199	0.93750	0.92333
0.2	0.96731	0.94674	0.93743	0.92624	0.91326	0.89836	0.88446
0.3	0.93358	0.91200	0.90252	0.89191	0.87844	0.86390	0.85047
0.4	0.89923	0.87807	0.86884	0.85795	0.84548	0.83144	0.81849
0.5	0.86577	0.84531	0.83639	0.82589	0.81386	0.80034	0.78787
0.6	0.83346	0.81375	0.80516	0.79504	0.78347	0.77044	0.75844
0.7	0.80234	0.78336	0.77509	0.76535	0.75421	0.74167	0.73011
0.8	0.77238	0.75410	0.74614	0.73677	0.72604	0.71397	0.70285
0.9	0.74353	0.72594	0.71827	0.70925	0.69892	0.68731	0.67660
1.0	0.71576	0.69883	0.69142	0.68276	0.67282	0.66164	0.65133
1.5	0.59167	0.57771	0.57158	0.56443	0.55621	0.54697	0.53844
2.0	0.48916	0.47758	0.47252	0.46661	0.45981	0.45217	0.44512
3.0	0.33429	0.32638	0.32292	0.31888	0.31423	0.30901	0.30419
4.0	0.22845	0.22304	0.22068	0.21792	0.21474	0.21117	0.20788
5.0	0.15612	0.15243	0.15081	0.14892	0.14675	0.14431	0.14207
6.0	0.10669	0.10417	0.10306	0.10177	0.10029	0.09862	0.09708
7.0	0.07291	0.07119	0.07043	0.06955	0.06854	0.06739	0.06635
8.0	0.04983	0.04865	0.04813	0.04753	0.04684	0.04606	0.04534
9.0	0.03405	0.03325	0.03289	0.03248	0.03201	0.03148	0.03099
10.0	0.02327	0.02272	0.02248	0.02220	0.02187	0.02151	0.02118
11.0	0.01590	0.01553	0.01536	0.01517	0.01495	0.01470	0.01447
12.0	0.01087	0.01061	0.01050	0.01037	0.01022	0.01005	0.00989
14.0	0.00508	....	....	....	....	....	0.00462
16.0	0.00237	....	....	....	....	....	0.00216
18.0	0.00111	....	......	....	....	....	0.00101

Cylinder. $HL = 0.4$

				Value of U when x/L is			
τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.05	N.E.	0.99114	0.98380	0.97135	0.95516	0.92842	0.89654
0.1	0.99032	0.96454	0.95073	0.93278	0.91051	0.88324	0.85163
0.2	0.94009	0.90201	0.88492	0.86456	0.84093	0.81412	0.78428
0.3	0.87856	0.84005	0.82325	0.80354	0.78099	0.75573	0.72788
0.4	0.81810	0.78163	0.76582	0.74731	0.72621	0.70264	0.67673
0.5	0.76120	0.72712	0.71237	0.69511	0.67546	0.65353	0.62942
0.6	0.70811	0.67639	0.66265	0.64660	0.62831	0.60790	0.58547
0.7	0.65869	0.62918	0.61640	0.60147	0.58446	0.56548	0.54461
0.8	0.61272	0.58527	0.57338	0.55949	0.54367	0.52601	0.50543
0.9	0.56996	0.54442	0.53336	0.52044	0.50572	0.48817	0.47124
1.0	0.53018	0.50642	0.49614	0.48412	0.47042	0.45514	0.43835
1.2	0.45875	0.43819	0.42831	0.41890	0.40705	0.39383	0.37929
1.4	0.39695	0.37916	0.37146	0.36247	0.35221	0.34077	0.32820
1.6	0.34347	0.32808	0.32142	0.31363	0.30476	0.29487	0.28398
1.8	0.29720	0.28389	0.27812	0.27138	0.26371	0.25514	0.24573
2.0	0.25705	0.24564	0.24065	0.23482	0.22818	0.22077	0.21262
3.0	0.12474	....	....	....	....	....	0.10314
4.0	0.06050	....	....	....	....	....	0.05003
5.0	0.02935	... ..	....	....	....	....	0.02426
6.0	0.01424	....	....	....	....	....	0.01177
7.0	0.00691	....	....	....	....	....	0.00571
8.0	0.00335	....	....	....	....	....	0.00277
9.0	0.00162	....	....	....	....	....	0.00134

Cylinder. $HL = 0.6$

			Value of U when x/L is				
τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.05	0.99935	0.98691	0.97603	0.95849	0.93264	0.89687	0.85088
0.10	0.98566	0.94888	0.92908	0.90362	0.87202	0.83389	0.78955
0.20	0.91429	0.86146	0.83791	0.80991	0.77770	0.74150	0.70109
0.30	0.82959	0.77782	0.75540	0.72915	0.69936	0.66616	0.62986
0.40	0.74897	0.70145	0.68099	0.65711	0.63011	0.60009	0.56738
0.50	0.67540	0.63240	0.61390	0.59233	0.56796	0.54089	0.51139
0.60	0.60892	0.57011	0.55342	0.53397	0.51199	0.48758	0.46098
0.70	0.54895	0.51394	0.49890	0.48136	0.46155	0.43954	0.41557
0.80	0.49486	0.46331	0.44975	0.43394	0.41608	0.39624	0.37463
0.90	0.44611	0.41767	0.40544	0.39119	0.37509	0.35721	0.33772
1.00	0.40216	0.37652	0.36550	0.35265	0.33814	0.32201	0.30445
1.20	0.32682	0.30599	0.29703	0.28659	0.27480	0.26169	0.24742
1.40	0.26560	0.24867	0.24139	0.23290	0.22332	0.21267	0.20107
1.60	0.21584	0.20208	0.19617	0.18927	0.18149	0.17283	0.16340
1.80	0.17541	0.16423	0.15942	0.15382	0.14749	0.14045	0.13279
2.00	0.14255	0.13346	0.12956	0.12500	0.11986	0.11414	0.10792
2.5	0.08487	....	....	....	....	....	0.06425
3.0	0.05053	....	....	....	....	....	0.03825
4.0	0.01791	....	....	....	....	....	0.01356
5.0	0.00635	....	....	....	....	....	0.00480
6.0	0.00225	....	....	....	....	....	0.00170

Cylinder. $HL = 0.80$

			Value of U when x/L is				
τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99904	0.99649	0.99127	0.97118	0.93551	0.87713
0.04	N.E.	0.99048	0.97971	0.96129	0.93128	0.88732	0.82836
0.05	0.99933	0.98314	0.96877	0.94648	0.91323	0.86756	0.80902
0.06	0.99784	0.97558	0.95673	0.93190	0.89633	0.84960	0.79167
0.08	0.99167	0.95525	0.93320	0.90378	0.86538	0.81770	0.76114
0.10	0.98119	0.93432	0.90917	0.87725	0.83742	0.78973	0.73451
0.15	0.94123	0.87989	0.85127	0.81685	0.77633	0.73001	0.67828
0.20	0.89142	0.82595	0.79678	0.76270	0.72341	0.67934	0.63090
0.30	0.78755	0.72522	0.69829	0.66721	0.63194	0.59289	0.55043
0.40	0.69145	0.63585	0.61194	0.58451	0.55343	0.51912	0.48191
0.50	0.60624	0.55731	0.53631	0.51221	0.48494	0.45486	0.42225
0.60	0.53136	0.48845	0.47002	0.44890	0.42499	0.39862	0.37004
0.70	0.46569	0.42808	0.41193	0.39342	0.37246	0.34935	0.32430
0.80	0.40814	0.37517	0.36102	0.34479	0.32643	0.30617	0.28422
0.90	0.35769	0.32880	0.31640	0.30218	0.28608	0.26833	0.24909
1.00	0.31348	0.28816	0.27729	0.26483	0.25072	0.23517	0.21831
1.20	0.24078	0.22133	0.21298	0.20341	0.19258	0.18063	0.16768
1.40	0.18494	0.17000	0.16359	0.15624	0.14792	0.13874	0.12879
1.50	0.16208	0.14899	0.14338	0.13693	0.12963	0.12159	0.11287
1.60	0.14205	0.13058	0.12565	0.12001	0.11309	0.10656	0.09892
1.80	0.10911	0.10029	0.09651	0.09217	0.08719	0.08185	0.07598
2.00	0.08381	0.07703	0.07413	0.07080	0.06703	0.06287	0.05836
3.00	0.02240	....	....	....	....	....	0.01560
4.00	0.00599	....	.... ..	....	....	....	0.00417
5.00	0.00160	....	....	....	....	....	0.00111

Cylinder. $HL = 1$
——Value of U when x/L is——

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.04	N.E.	0.99013	0.97748	0.97638	0.91801	0.86467	0.79388
0.05	N.E.	0.98114	0.96416	0.93687	0.89640	0.84128	0.77338
0.06	0.99937	0.97070	0.95209	0.91929	0.87630	0.82022	0.75133
0.08	0.99179	0.94737	0.92134	0.88353	0.83982	0.78312	0.71637
0.10	0.97904	0.92337	0.89278	0.85423	0.80716	0.75080	0.68635
0.20	0.87208	0.79536	0.76197	0.72227	0.67709	0.62659	0.57162
0.30	0.75176	0.68071	0.65064	0.61545	0.57598	0.53242	0.48553
0.40	0.64340	0.58167	0.55568	0.52539	0.48151	0.45423	0.41491
0.50	0.54977	0.49686	0.47461	0.44868	0.41971	0.38786	0.35366
0.60	0.46963	0.42438	0.40537	0.38321	0.35846	0.33125	0.30204
0.70	0.40111	0.36246	0.34623	0.32731	0.30616	0.28292	0.25797
0.80	0.34259	0.30958	0.29572	0.27955	0.26150	0.24164	0.22034
0.90	0.29261	0.26442	0.25257	0.23877	0.22855	0.20639	0.18819
1.0	0.24992	0.22584	0.21424	0.20393	0.19032	0.17628	0.16073
1.2	0.18231	0.16475	0.15737	0.14877	0.13916	0.12860	0.11726
1.4	0.13300	0.12018	0.11480	0.10853	0.10152	0.09381	0.08554
1.6	0.09702	0.08767	0.08375	0.07917	0.07406	0.06843	0.06237
1.8	0.07078	0.06396	0.06109	0.05910	0.05402	0.04992	0.04550
2.0	0.05163	0.04666	0.04457	0.04213	0.03941	0.03642	0.03319
3.0	0.01067						0.00686
4.0	0.00220						0.00142

Cylinder. $HL = 2$
——Value of U when x/L is——

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.04	0.99616	0.97812	0.95515	0.91569	0.85210	0.76204	0.64513
0.05	0.99748	0.96246	0.93229	0.88603	0.81709	0.72597	0.61304
0.06	0.99419	0.94464	0.90894	0.85695	0.78545	0.69450	0.58536
0.08	0.98101	0.90584	0.86198	0.80403	0.73165	0.64121	0.53906
0.10	0.95921	0.86560	0.81725	0.75655	0.68264	0.59691	0.50099
0.15	0.88075	0.76738	0.71627	0.65592	0.58664	0.50980	0.42687
0.2	0.78992	0.67738	0.62902	0.57321	0.51064	0.44152	0.37016
0.3	0.61929	0.52557	0.48753	0.44183	0.39255	0.33956	0.28383
0.4	0.48072	0.40711	0.37651	0.34178	0.30350	0.26244	0.21921
0.5	0.37240	0.31525	0.29150	0.26336	0.23493	0.20312	0.16974
0.6	0.28834	0.24409	0.22570	0.20485	0.18230	0.15762	0.13142
0.7	0.22329	0.18900	0.17436	0.15861	0.14083	0.12176	0.10176
0.8	0.17289	0.14634	0.13531	0.12281	0.10904	0.09428	0.07879
0.9	0.13387	0.11331	0.10472	0.09509	0.08442	0.07300	0.06101
1.0	0.10365	0.08773	0.08112	0.07363	0.06537	0.05652	0.04724
1.5	0.02884	0.02441	0.02255	0.02049	0.01823	0.01372	0.01315
2.0	0.00803	0.00679	0.00628	0.00570	0.00506	0.00438	0.00366
2.2	0.00481	....	....	....	....	....	0.00219
2.4	0.00289	....	....	....	....	....	0.00131
2.6	0.00173	....	....	....	....	....	0.00079

Cylinder. $HL = 2.4$
——Value of U when x/L is——

	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99677	0.99072	0.97074	0.92295	0.83355	0.69204
0.04	0.99940	0.97474	0.94821	0.90230	0.82989	0.72883	0.59799
0.05	0.99760	0.95681	0.92230	0.86866	0.79108	0.68950	0.56409
0.06	0.99374	0.93660	0.89579	0.83677	0.75634	0.65557	0.53518
0.08	0.97591	0.89300	0.84370	0.77859	0.69623	0.59889	0.48746
0.10	0.95358	0.84830	0.79449	0.72704	0.64550	0.55248	0.44884
0.15	0.86579	0.74100	0.69529	0.61971	0.54472	0.46299	0.37512
0.20	0.76618	0.64459	0.59282	0.53327	0.46667	0.39543	0.32000
0.30	0.58385	0.48584	0.44521	0.39914	0.34828	0.29453	0.23816
0.40	0.44036	0.36566	0.33484	0.29998	0.26160	0.22115	0.17879
0.50	0.33147	0.27512	0.25189	0.22563	0.19676	0.16632	0.13445
0.60	0.24941	0.20699	0.18951	0.16975	0.14802	0.12511	0.10115
0.70	0.18764	0.15572	0.14258	0.12771	0.11136	0.09413	0.07610
0.80	0.14117	0.11716	0.10727	0.09608	0.08378	0.07082	0.05725
0.90	0.10621	0.08814	0.08070	0.07229	0.06303	0.05328	0.04307
1.0	0.07991	0.06631	0.06071	0.05439	0.04742	0.04008	0.03241
1.2	0.04523	0.03754	0.03437	0.03078	0.02684	0.02269	0.01834
1.4	0.02560	0.02125	0.01945	0.01742	0.01519	0.01284	0.01038
1.5	0.01926	0.01598	0.01463	0.01311	0.01143	0.00966	0.00781
1.6	0.01449	0.01203	0.01101	0.00986	0.00860	0.00727	0.00588
1.8	0.00820	0.00680	0.00623	0.00558	0.00487	0.00411	0.00332
2.0	0.00464	0.00385	0.00353	0.00316	0.00276	0.00233	0.00188
2.2	0.00263						0.00107
2.4	0.00149						0.00060

Cylinder. HL = 4

Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99931	0.99266	0.98504	0.95586	0.88710	0.75858	0.56290
0.04	0.99786	0.96237	0.92472	0.87053	0.76224	0.62607	0.45661
0.05	0.99532	0.93744	0.88942	0.81611	0.71299	0.57968	0.42098
0.06	0.98970	0.90998	0.85430	0.77522	0.66896	0.54095	0.39164
0.08	0.96823	0.85385	0.78760	0.70336	0.59923	0.47986	0.34523
0.10	0.93439	0.79640	0.72701	0.64204	0.54232	0.43031	0.30937
0.15	0.82047	0.66635	0.59923	0.52163	0.43513	0.34202	0.24503
0.20	0.69869	0.55632	0.49709	0.43006	0.35686	0.27943	0.19988
0.30	0.49164	0.38697	0.34445	0.30063	0.24560	0.19188	0.13713
0.40	0.34242	0.26898	0.23927	0.20612	0.17041	0.13307	0.09509
0.50	0.23806	0.18695	0.16627	0.14322	0.11840	0.09275	0.06606
0.60	0.16547	0.12993	0.11556	0.09953	0.08228	0.06425	0.04591
0.70	0.11500	0.09030	0.08031	0.06918	0.05718	0.04465	0.03190
0.80	0.07992	0.06275	0.05582	0.04808	0.03974	0.03103	0.02217
0.90	0.05555	0.04361	0.03879	0.03341	0.02762	0.02157	0.01541
1.00	0.03860	0.03031	0.02696	0.02322	0.01920	0.01499	0.01071
1.20	0.01865	0.01464	0.01302	0.01122	0.00927	0.00724	0.00517
1.40	0.00900	0.00707	0.00629	0.00542	0.00448	0.00350	0.00250
1.50	0.00626	0.00491	0.00437	0.00377	0.00311	0.00243	0.00174
1.60	0.00435	....	....	....	....	....	0.00121
1.80	0.00210	....	....	....	....	....	0.00058

Cylinder. HL = 6

Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99563	0.97961	0.94155	0.85177	0.68957	0.45024
0.04	0.99886	0.95236	0.90426	0.82415	0.70338	0.53841	0.34566
0.05	0.99533	0.92124	0.86149	0.77195	0.64826	0.49100	0.31283
0.06	0.98805	0.88792	0.81994	0.72512	0.60172	0.45226	0.28659
0.08	0.96084	0.82027	0.74336	0.64526	0.52677	0.39172	0.24654
0.10	0.91835	0.75542	0.67591	0.57958	0.46824	0.34563	0.21673
0.15	0.78602	0.61289	0.53792	0.45493	0.36257	0.26496	0.16543
0.20	0.65034	0.49689	0.43421	0.36409	0.28861	0.21007	0.13093
0.30	0.43206	0.32655	0.28432	0.23759	0.18821	0.13637	0.08492
0.40	0.28441	0.21460	0.18675	0.15597	0.12322	0.08945	0.05568
0.50	0.18696	0.14103	0.12271	0.10248	0.08095	0.05876	0.03659
0.60	0.12286	0.09268	0.08064	0.06735	0.05320	0.03862	0.02405
0.70	0.08073	0.06091	0.05300	0.04426	0.03496	0.02538	0.01580
0.80	0.05306	0.04003	0.03491	0.02908	0.02297	0.01668	0.01036
0.90	0.03487	0.02630	0.02289	0.01911	0.01510	0.01096	0.00681
1.00	0.02291	0.01729	0.01504	0.01256	0.00992	0.00720	0.00447
1.2	0.00900	0.00747	0.00649	0.00543	0.00428	0.00311	0.00193
1.4	0.00427	0.00322	0.00280	0.00234	0.00185	0.00134	0.00084
1.5	0.00281	0.00212	0.00184	0.00154	0.00121	0.00088	0.00055
1.6	0.00185	....	....	....	....	....	0.00036

Cylinder. HL = 7

Value of U when x/L is

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	0.99959	0.99324	0.97615	0.93826	0.83762	0.66188	0.40756
0.04	0.99747	0.94693	0.89518	0.81035	0.68124	0.51031	0.30659
0.05	0.99372	0.91373	0.84974	0.75520	0.62446	0.46815	0.27570
0.06	0.98581	0.87840	0.80602	0.70636	0.57698	0.42246	0.25126
0.08	0.95655	0.80906	0.72660	0.62423	0.50140	0.36249	0.21442
0.10	0.91251	0.74063	0.65733	0.55758	0.44308	0.31780	0.18736
0.15	0.77284	0.59423	0.51868	0.43284	0.33921	0.24084	0.14135
0.20	0.63320	0.47709	0.41371	0.34310	0.26744	0.18913	0.11082
0.25	0.51239	0.38315	0.33033	0.27419	0.21329	0.15070	0.08819
0.30	0.41272	0.30772	0.26560	0.21982	0.17085	0.12057	0.07058
0.35	0.33186	0.24717	0.21342	0.17644	0.13710	0.09640	0.05662
0.40	0.26670	0.19853	0.17145	0.14167	0.11008	0.07765	0.04545
0.45	0.21423	0.15947	0.13773	0.11378	0.08840	0.06215	0.03650
0.50	0.17209	0.12809	0.11063	0.09139	0.07100	0.05009	0.02932
0.60	0.11103	0.08264	0.07137	0.05896	0.04581	0.03232	0.01891
0.70	0.07163	0.05332	0.04605	0.03804	0.02955	0.02085	0.01220
0.80	0.04622	0.03440	0.02971	0.02454	0.01907	0.01345	0.00787
0.90	0.02982	0.02219	0.01917	0.01583	0.01230	0.00868	0.00508
1.0	0.01924	0.01432	0.01237	0.01022	0.00794	0.00560	0.00328
1.2	0.00801	....	....	....	....	....	0.00136
1.4	0.00333	....	....	....	....	....	0.00057
1.5	0.00215	....	....	....	....	....	0.00037

Cylinder. $HL = 10$

τ	Value of U when x/L is						
	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99176	0.97753	0.91991	0.80419	0.60102	0.31338
0.04	0.99758	0.93678	0.88088	0.77742	0.63307	0.44568	0.22635
0.05	0.99280	0.89879	0.83028	0.71805	0.57391	0.39830	0.20094
0.06	0.98335	0.85918	0.78199	0.66643	0.52551	0.36100	0.18125
0.08	0.94931	0.78149	0.69637	0.58135	0.45027	0.30530	0.15239
0.10	0.89935	0.70956	0.62362	0.51380	0.39354	0.26471	0.13165
0.15	0.74686	0.55776	0.48176	0.39044	0.29492	0.19641	0.09723
0.20	0.59981	0.43926	0.37704	0.30364	0.22828	0.15049	0.07485
0.30	0.37361	0.27301	0.23362	0.18765	0.14068	0.09318	0.04601
0.40	0.23430	0.16977	0.14522	0.11660	0.08740	0.05787	0.02858
0.50	0.14574	0.10559	0.09031	0.07251	0.05434	0.03598	0.01777
0.60	0.09064	0.06567	0.05617	0.04510	0.03380	0.02238	0.01105
0.70	0.05637	0.04084	0.03493	0.02805	0.02102	0.01392	0.00687
0.80	0.03506	0.02540	0.02173	0.01744	0.01307	0.00866	0.00427
0.90	0.02180	0.01580	0.01351	0.01085	0.00813	0.00539	0.00266
1.00	0.01356	0.00982	0.00840	0.00675	0.00506	0.00335	0.00165
1.20	0.00524	0.00380	0.00325	0.00261	0.00196	0.00130	0.00064
1.40	0.00203	0.00147	0.00126	0.00101	0.00076	0.00050	0.00025
1.50	0.00126	0.00092	0.00078	0.00063	0.00047	0.00031	0.00015

Cylinder. $HL = 15$

τ	Value of U when x/L is						
	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.99043	0.98119	0.90815	0.76901	0.53639	0.22266
0.04	0.99810	0.92769	0.85927	0.74714	0.58703	0.38429	0.15520
0.05	0.99202	0.88494	0.80285	0.68398	0.52708	0.33969	0.13628
0.06	0.98104	0.84133	0.75125	0.63009	0.47892	0.30549	0.12200
0.08	0.94255	0.75779	0.66183	0.54325	0.40559	0.25528	0.10136
0.10	0.88723	0.68210	0.58744	0.47597	0.35137	0.21938	0.08684
0.15	0.72387	0.52651	0.44557	0.35520	0.25893	0.16006	0.06311
0.20	0.57119	0.40740	0.34310	0.25995	0.19730	0.12162	0.04790
0.25	0.44581	0.31623	0.26560	0.20080	0.15223	0.09372	0.03688
0.30	0.34668	0.24545	0.20596	0.15562	0.11792	0.07256	0.02852
0.35	0.26927	0.19052	0.15982	0.12069	0.09146	0.05627	0.02214
0.40	0.20906	0.14788	0.12403	0.09808	0.07097	0.04365	0.01718
0.45	0.16224	0.11479	0.09628	0.07613	0.05509	0.03389	0.01334
0.50	0.12654	0.08910	0.07473	0.05909	0.04276	0.02631	0.01035
0.55	0.09777	0.06916	0.05801	0.04587	0.03319	0.02042	0.00803
0.60	0.07589	0.05368	0.04502	0.03560	0.02576	0.01585	0.00623
0.70	0.04571	....	....	....	....	....	0.00376
0.80	0.02754	....	....	....	....	....	0.00226
0.90	0.01659	..... .	....	....	....	....	0.00136
1.00	0.01000	....	....	....	....	....	0.00082
1.20	0.00363	.... ..	....	....	....	....	0.00030
1.40	0.00132	....	....	....	....	....	0.00011

Cylinder. HL = 30

——————Value of *U* when *x/L* is——————

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	N.E.	N.E.	N.E.	N.E.	N.E.	N.E.
0.04	0.99839	0.91186	0.83102	0.70461	0.52909	0.31430	0.07575
0.05	0.99134	0.86397	0.77077	0.63922	0.47062	0.27539	0.06625
0.06	0.97798	0.82559	0.71642	0.58468	0.42457	0.24589	0.05915
0.08	0.93394	0.72980	0.62416	0.49867	0.35575	0.20329	0.04904
0.10	0.87109	0.64907	0.54908	0.43305	0.30571	0.17331	0.04193
0.15	0.69563	0.49139	0.40889	0.31470	0.22158	0.12442	0.03023
0.20	0.54186	0.37398	0.30956	0.23959	0.16628	0.09310	0.02265
0.25	0.41263	0.28513	0.23562	0.18615	0.12621	0.07059	0.01719
0.30	0.31547	0.21751	0.17964	0.13863	0.09614	0.05376	0.01308
0.35	0.24074	0.16596	0.13703	0.10574	0.07333	0.04099	0.00998
0.40	0.18369	0.12663	0.10456	0.08067	0.05594	0.03127	0.00762
0.45	0.14017	0.09662	0.07975	0.06156	0.04268	0.02386	0.00581
0.50	0.10695	0.07373	0.06088	0.04697	0.03257	0.01821	0.00443
0.60	0.06227	0.04292	0.03545	0.02735	0.01896	0.01060	0.00258
0.70	0.03626	0.02499	0.02064	0.01592	0.01104	0.00617	0.00150
0.80	0.02111	0.01455	0.01202	0.00927	0.00643	0.00359	0.00088
0.90	0.01229	0.00847	0.00700	0.00540	0.00357	0.00209	0.00051
1.00	0.00716						0.00030
1.20	0.00243						0.00010

Cylinder, HL = ∞

——————Value of *U* when *x/L* is——————

τ	0.0	0.5	0.6	0.7	0.8	0.9	1.0
0.02	N.E.	0.98429	0.94081	0.83942	0.63713	0.34802	0
0.04	0.99628	0.88906	0.79433	0.65141	0.46025	0.23473	0
0.05	0.98434	0.83555	0.73000	0.58562	0.40630	0.20449	0
0.06	0.97054	0.78396	0.67357	0.53191	0.36437	0.18181	0
0.08	0.91776	0.69061	0.58016	0.44887	0.30249	0.14938	0
0.10	0.84836	0.61025	0.50576	0.38661	0.25805	0.12669	0
0.15	0.66183	0.45262	0.32229	0.27866	0.18402	0.08974	0
0.20	0.50150	0.33797	0.27467	0.20633	0.13582	0.06611	0
0.30	0.28271	0.18934	0.15362	0.11523	0.07576	0.03684	0
0.40	0.15850	0.10618	0.08613	0.06460	0.04246	0.02065	0
0.50	0.08889	0.05955	0.04831	0.03623	0.02382	0.01158	0
0.60	0.04985	0.03340	0.02709	0.02032	0.01336	0.00649	0
0.70	0.02796	0.01874	0.01519	0.01140	0.00749	0.00364	0
0.80	0.01568	0.01051	0.00852	0.00639	0.00420	0.00205	0
0.90	0.00880	0.00589	0.00478	0.00358	0.00236	0.00115	0
1.00	0.00494	0.00331	0.00268	0.00201	0.00132	0.00064	0
1.20	0.00155	0.00104	0.00084	0.00063	0.00042	0.00020	0

Appendix II
Dimensions and Conversion Factors for Use in Heat Flow Calculations

Length (l)   1 inch $=$ 2.54 cm.
                    1 ft $=$ 30.48 cm.

Mass (m)   1 lb $=$ 453.59 grams

Energy (ml^2t^{-2}) 1 gram-calorie $(15°)$ $=$ 4.185 joules
                                              $=$ 4.185 watt. sec.
                                              $=$ 0.003968 Btu $(60°)$
                    1 Btu $(60°)$ $=$ 1055 watt. sec.

Power (ml^2t^{-3}) 1 horse-power $=$ 550 ft. lb. per sec.
                                              $=$ 745.7 watts
                    1 watt $=$ 0.2389 cals. per sec.

Density   1 lb per cu.in. $=$ 27.680 grams per cc.

Thermal Conductivity $(T^{-1}mlt^{-3})$
            1 cal $(15°)$ cm^{-1}.sec^{-1}.$°$C^{-1} $=$ 4.185 joules.cm^{-1}.sec^{-1}.$°$C^{-1}
                                        $=$ 4.185 watt.cm^{-1}.$°$C^{-1}
                                        $=$ 2903 Btu $(60°)$.ft^{-2}.hr^{-1}.in.$°$F^{-1}
                                        $=$ 48.38 Btu $(60°)$ ft^{-2}.min^{-1}.in.$°$F^{-1}
                                        $=$ 0.8064 Btu $(60°)$ ft^{-2}.sec.$^{-1}$.in.$°$F^{-1}
       1 watt.cm^{-1}.$°$C^{-1} $=$ 693.5 Btu $(60°)$ ft^{-2}.hr^{-1}.in.$°$F^{-1}

Thermal Diffusivity (l^2t^{-1}) 1 m^2.hr^{-1} $=$ 2.7778 cm^2.sec^{-1}
                                     1 ft^2.sec^{-1} $=$ 929.03 cm^2.sec^{-1}
                                     1 ft^2.hr^{-1} $=$ 0.25806 cm^2.sec^{-1}
                                     1 in^2.sec^{-1} $=$ 6.45 cm^2.sec^{-1}
                                     1 in^2.min^{-1} $=$ 0.1075 cm^2.sec^{-1}
                                     1 in^2.hr^{-1} $=$ 0.00179 cm^2.sec^{-1}

Heat Flow (mt^{-2}) 1 cal.per cm^2 $=$ 3.687 Btu per ft^2.

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